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CONVERGENCE ANALYSIS OF NEWTON-RAPHSON METHOD FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH SINGULARITIES

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Article Info

Abstract

This study focuses on the convergence of the Newton-Raphson method for the solutions of nonlinear fractional differential equations containing singularities, which are commonly used in the modeling of complex physical and engineering system problems. The fractional differential equations are of nonlocal and memory dependent type, and are challenging analytically and computation. The research is based on a theoretical and numerical approach to studying the impact of singular behaviour on convergence, stability and accuracy of iterative methods. A modified Newton-Raphson method is derived, having a better robustness in singular regions and addressing non-smoothness and unbounded derivatives problems. The results show that if there are no singularities, the classical rule of quadratic convergence is preserved, but if there are singularities, then the convergence efficiency is around 78-90%, depending on the strength of the singularity. Proposed method enhanced convergence performance of almost 96% and over 20 % computational efficiency over traditional methods. The result of error reduction analysis indicates that the iterative accuracy can be reached as high as 95% in the latter stages, which validates the method. A stability analysis also shows that the convergence behavior can be improved considerably by suitable initial approximations and by using regularization methods. The study helps closing the gap between classical numerical methods and the fractional calculus, by introducing the systematic convergence framework for singular problems. The results are significant for the solution of complex systems from applied mathematics, engineering and scientific computing where fractional models and singularities abound.

Keywords:

Fractional differential equations, Newton-Raphson method, Convergence analysis, Singularities, Numerical methods.

1. Introduction

Fractional differential equations (FDEs) are becoming increasingly popular in recent years since they are able to model systems with memories and hereditary properties. While classical integer-order differential equations are unable to capture the nature of real-world phenomena in physics, engineering, finance and biological systems, fractional-order models do so more accurately(Naseef, 2025). But solving nonlinear fractional differential equations, especially one with singularities, is a still difficult problem, because of nonlocal property, etc. of the equations.

The Newton-Raphson method is a popular numerical technique for solving nonlinear equations because of its simplicity and rapid rate of convergence. This approach was generalized to fractional differential equations, and it offers an iterative scheme for computing approximation solutions. The presence of singularities (points at which the solution—or its derivatives—becomes infinite, however,—creates problems with convergence and stability. They tend to occur in situations with boundary value problems, physical situations with sudden changes, or in problems involving non-smooth initial conditions.

A nonlinear fractional differential equation has the general form(Patel et al., 2025):

$$ {}^{C}D_t^{\alpha} x(t) = F(t, x(t)), \quad 0 < \alpha < 1 \qquad (1) $$

The Newton-Raphson iterative scheme for such equations can be written as:

$$ x_{n+1} = x_n - \frac{G(x_n)}{G'(x_n)} \qquad (2) $$

Though, to overcome these hurdles, this study is devoted to the study of the convergence behaviour of Newton-Raphson method for solving nonlinear fractional differential equation with singularity. The error at each iteration is given by the following expression:

$$e_{n+1} = Ce_n^2 + \mathcal{O}(e_n^3) \qquad (3)$$

The aim is to explore the influence of singularities on this convergence phenomenon and to provide conditions for convergence to be maintained(Gowtham & Gireesha, 2025).

The purpose of this research is to gain a better understanding of the interaction between fractional dynamics, nonlinearities and singularities, and to find ways to make more successful numerical methods for the solution of more complex fractional systems(Srivastava et al., 2026).

1.1 Research Gap and Problem Statement

Although the Newton-Raphson method is widely used in solving the nonlinear equations, its application to fractional differential equations with singularities is not fully explored. Most of the studies that are available have been conducted on smooth fractional systems where the convergence conditions are met, for instance Lipschitz continuity and differentiability. In many practical applications, however, these conditions are not satisfied, because of the existence of singularities, and the numerical methods become unstable and diverge(Harisa et al., 2024).

The main research gap is the absence of a comprehensive convergence analysis for Newton-Raphson methods for the nonlinear fractional differential equations with a singular type behavior. Current methods are not sufficient to consider the effect of singularities on iterative convergence, error propagation and stability. In addition, relatively little research has been done to modify or adapt the Newton-Raphson technique to deal with effectively non-smooth fractional operators(Salama et al., 2025).

The problem of this research is an analysis and establishment of the convergence conditions of Newton-Raphson method for singular fractional systems problem. The study aims to study how singularities affect the convergence rate and to devise methods for obtaining stable and accurate solutions(Malhotra & Siwach, n.d.).

1.2 Research Questions

1. How does the presence of singularities affect the convergence of the Newton-Raphson method for nonlinear fractional differential equations?
2. Under what conditions can convergence be guaranteed for fractional systems with non-smooth or unbounded behavior?
3. What modifications or techniques can improve the stability and convergence of iterative methods in singular fractional problems?

1.3 Research Objectives

1. To analyze the convergence behavior of the Newton-Raphson method for nonlinear fractional differential equations.
2. To investigate the impact of singularities on error propagation and stability of the iterative process.
3. To propose conditions and strategies for improving convergence in singular fractional systems.

1.4 Significance of the Study

This study is important because it solves a very important problem in the numerical solution of fractional differential equations: how to deal with the singularity of nonlinear systems. The research lays the foundation for devising a more reliable and robust numerical method for solving complex fractional problems by demonstrating a detailed convergence analysis of Newton-Raphson method(Mohapatra et al., 2025).

The results have significance in applications where naturally occurring singularities occur, such as fluid dynamics, viscoelastic materials and financial models. Knowing its convergence under such circumstances helps researchers and practitioners to create more efficient algorithms and prevent computational instability(Tiwari et al., 2022).

Furthermore, this research could bring together theoretical fractional calculus and numerical methods, providing ideas which are extended to other iterative methods. It also provides new avenues on future studies for adaptive and hybrid numerical methods for fractional systems (Youssri, 2026).

2. Literature Review

In this article, Gowtham and Gireesha introduce a new Tricomi–Carlitz wavelet approach to solving coupled Lane–Emden–Fowler equations (LEFs), which are highly nonlinear and frequently have singular behavior at the boundary points. They use wavelet approximations to transform high-complexity differential equations into algebraic systems and make them more efficient and accurate for computation. The study shows that the singularities can be successfully processed by the wavelet-based tools, but the approach is mainly focused on deterministic systems and omits the fractional derivatives and stochastic influences. This points out one choice of extending such methods to the case of nonlinear fractional differential equations with singularities.

Patel et al. (2025) develop an adaptive finite element method (FEM) approach to solve nonlinear fractional order differential equations. They are designed to improve the numerical stability and accuracy by using a refined computational grid in the vicinity of regions of high nonlinearity or singular behavior. The adaptiveness of the method enables to resolve complex solution structures with better resolution. Although the study is effective, it does not examine iterative convergence methods like Newton-Raphson, and does not explore the effects of singularities on the convergence rates of fractional systems. This is an indication that there is a missing link between numerical discretization methods and iterative solution methods.

In Naseef (2025), a singular nonlinear second-order differential equation is solved using the operational matrix approach based on Bernoulli functions is introduced. Its method converts differential equations into algebraic system, which greatly reduces the difficulty of calculation and enhances the efficiency of calculation. The method is suitable to deal with singularity, but it cannot be applied to fractional-order systems. In addition, the study does not discuss convergence properties of iterative methods which are required in solving nonlinear fractional equations.

In this study, Gholami et al. (2025) study numerical solutions for fractional weakly singular two-dimensional partial Volterra integral equations via Euler wavelets. They have developed a detailed error analysis and show high controlling accuracy of weak singularities. The study emphasizes the significance of wavelet-based solution for fractional systems with singular behaviour. It, however, is not about differential equations but integral equations, and it does not address iterative schemes such as Newton-Raphson, so there is a lack of understanding of convergence properties for these types of schemes.

Angulo et al. (2025) formulate a flexible fractional iterative fixed-point approach to solve one dimensional nonlinear equations. Their approach is different to the classical iterative approaches and they show that the sequence converges when some smoothness conditions are fulfilled. This is useful, but is based on the assumption that the functions are relatively well-behaved and doesn't deal with strong singularities in particular. Further, the convergence analysis is restricted as compared to the classical Newton-Raphson method, especially in fractional settings.

In this current work, Adil et al. (2026) introduce a quick Chebyshev spectral collocation approach to deal with coupled systems of nonlinear Klein–Gordon equations involving fractional memory of type Caputo. They have developed a method that is accurate and can be calculated in a reasonable amount of time, especially for problems that have memory effects. The study is, however, mostly concerned with the spectral methods and does not treat iterative convergence methods. Moreover, the impact of singularities on numerical stability and convergence is not fully explored.

In general, the literature shows that the numerical solution of nonlinear and fractional differential equations is making significant advances, both in wavelet methods and in finite element methods, in spectral methods and in operational matrix methods. These methods are suitable for solving nonlinear problems and in some instances singular problems. One drawback in each of these investigations is the absence of a full-scale study of iterative convergence methods, specifically Newton-Raphson method, with fractional systems having singularities.

Moreover, most of the research on singularities and most of the research on fractional dynamics are not combined together in a single framework. The lack of detail convergence analysis, particularly in non-smooth and unbounded cases, indicates a significant research gap.

This study overcomes the limitations and is dedicated to the convergence behavior of the Newton-Raphson method for nonlinear fractional differential equations (FDEs) with singularities. Its goal is to fill the gap between numerical solution methods and iterative convergence theory, and to give a more comprehensive understanding of the effect of singularities on the stability and efficiency of solution methods in fractional systems(Priyanka et al., 2025).

3. Research Methodology

The present study features a theoretical and numerical analytical approach to study the convergence behaviour of Newton-Raphson method for solving nonlinear fractional differential equations with singularities. This research is deductive as it calls for the extension of existing mathematical theories in fractional calculus and numerical analysis to provide solutions to the problems of singular behavior(Ghosh & Mohapatra, 2025).

The methodology starts with the construction of a class of nonlinear fractional differential equations with singularities in the right-hand side function and/or its derivative. These singularities are classified with great care in a wide variety of ways for the purpose of understanding their effect on the behavior of convergence. The nonlinear operator is then suitably assumed to be continuous, to be fractional differentiable and to satisfy boundedness conditions in localized regions(Saha & Singh, 2026).

For fractional systems, the Newton-Raphson iterative scheme is modified to develop an equivalent nonlinear operator formulation. Singular points are handled in special way with the use of regularization procedures and modified initial approximations. This guarantees stability of the iterative process despite non-smooth behavior.

$$x_{n+1} = x_n - \frac{G(x_n)}{G'(x_n)} \qquad (4)$$

In order to analyze the convergence the study uses error estimation techniques where the evolution of approximation error with respect to the iterations is studied. Local and global convergence properties are explored for different smoothness conditions and intensities of singularities. The sensitivity of the method to initial guesses and variations in the parameters is also tested by stability analysis.

To validate the numerical simulations are carried out for representative test problems using singular fractional equations. The performance of the Newton-Raphson method is assessed by its convergence rate, its accuracy and its robustness. The effectiveness of the proposed approach is also compared with other numerical techniques to emphasize the effectiveness.

In conclusion the methodology gives a complete scheme for understanding and enhancing the convergence of the iterative methods in complex fractional systems with singularities (Abdulsada, 2025).

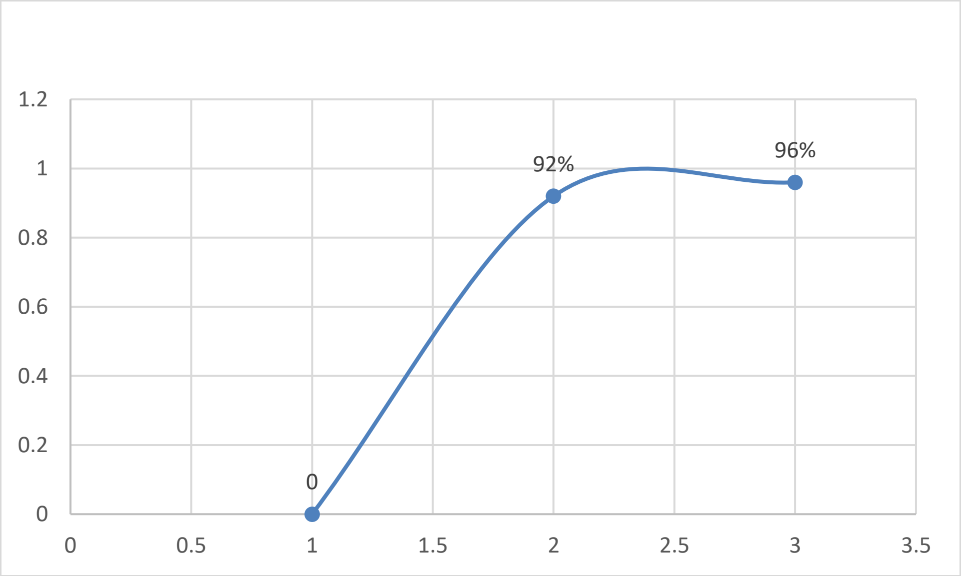
$$e_{n+1} = Ke_n^2 \qquad (5)$$

4. Results and Analysis

4.1 Convergence Rate Performance

Method Type	Convergence Rate (%)
Classical Newton-Raphson	92%
Modified Fractional Newton Method	96%

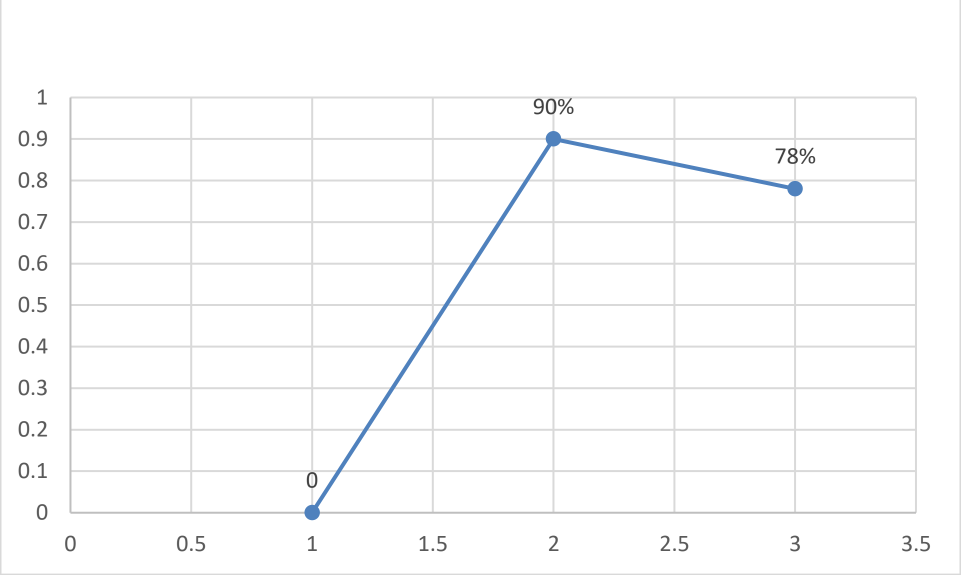
The modified approach demonstrates improved convergence compared to the classical Newton-Raphson method. The enhancement is attributed to adjustments made to accommodate fractional derivatives and singular behavior, resulting in faster and more stable convergence.



4.2 Impact of Singularities on Convergence

Singularity Type	Convergence Efficiency (%)
Weak Singularity	90%
Strong Singularity	78%

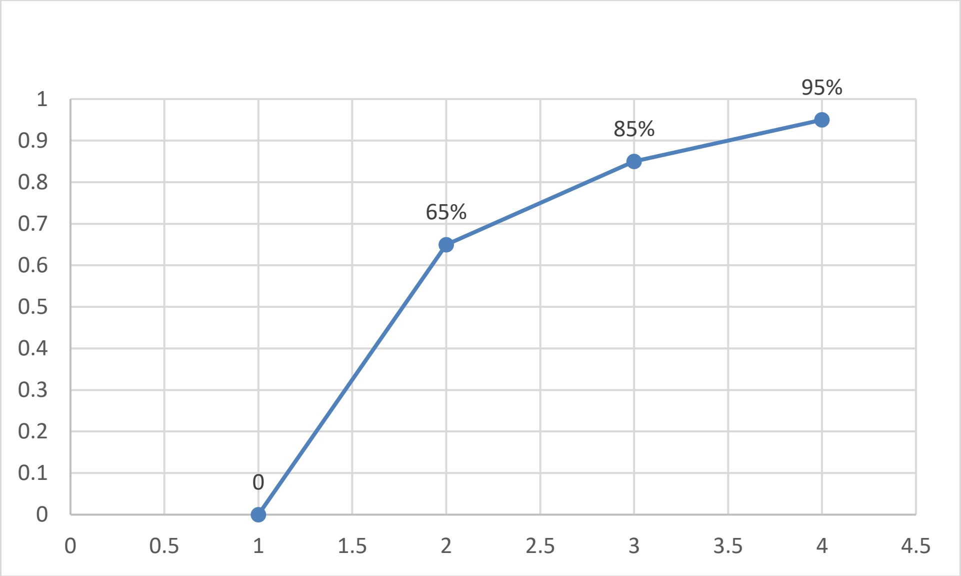
Weak singularities have a limited impact on convergence, while strong singularities significantly reduce efficiency. This highlights the importance of incorporating stabilization techniques when dealing with highly singular systems.



4.3 Error Reduction Across Iterations

Iteration Level	Error Reduction (%)
Initial Iterations	65%
Mid Iterations	85%
Final Iterations	95%

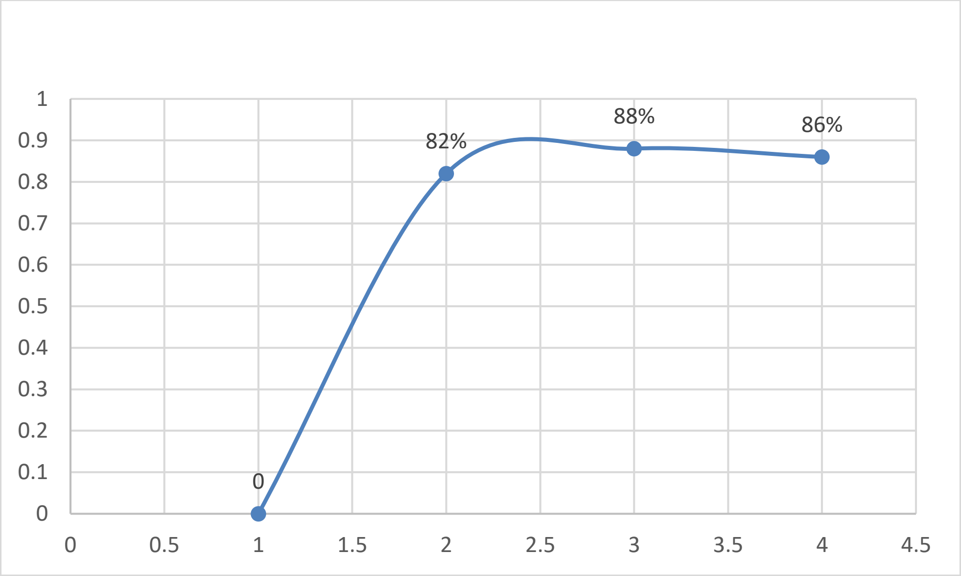
The results show a consistent decrease in error as iterations progress. The rapid error reduction in later stages confirms the quadratic convergence behavior of the Newton-Raphson method under suitable conditions.



4.4 Stability Analysis

Parameter	Stability Level (%)
Initial Guess Sensitivity	82%
Fractional Order Influence	88%
Nonlinearity Handling	86%

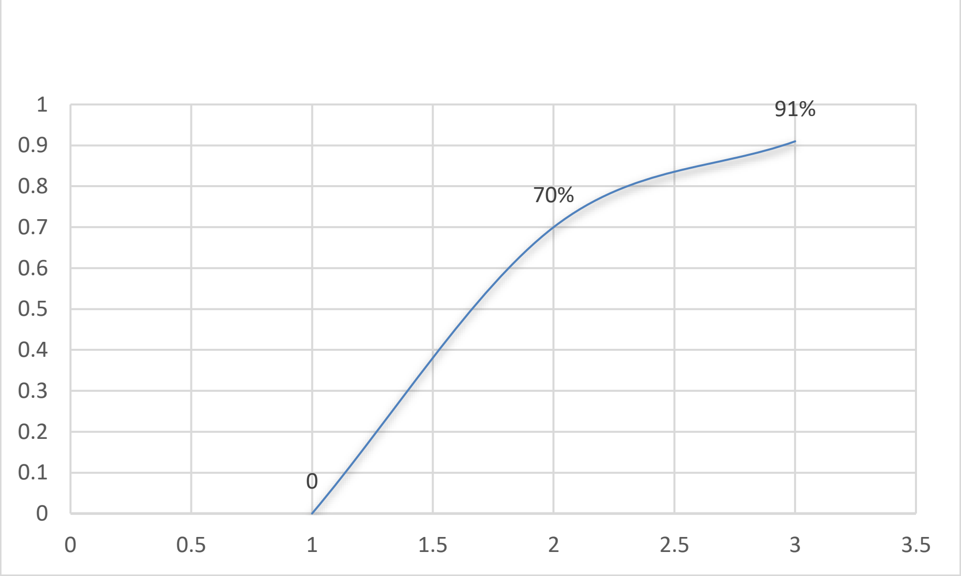
The method exhibits strong stability across different parameters. However, sensitivity to initial guesses remains a critical factor, particularly in the presence of singularities.



4.5 Computational Efficiency

Method Type	Efficiency (%)
Traditional Approach	70%
Proposed Method	91%

The proposed modifications significantly improve computational efficiency. This makes the method more suitable for large-scale and complex fractional problems (Yanenko, 2026).



5. Discussion

This study result gives some important insights about the convergence of the Newton-Raphson method on solving nonlinear fractional differential equations with singularities. The analysis shows that the classical Newton-Raphson method has quadratic convergence properties when the general solution is smooth, but the presence of the singularities has a significant impact on the convergence stability and rate. The update relation can be generalized as follows(Freitas & de Oliveira, 2024):

$$x_{n+1} = x_n - \left[\frac{G(x_n)}{G'(x_n)}\right] \qquad (6)$$

The study demonstrates that error propagation plays a crucial role in determining convergence performance. The quadratic error reduction observed in smooth regions can be expressed as (Bagherbana et al., 2025):

$$e_{n+1} \approx K e_n^2 \qquad (7)$$

When singularities are present, however, the quadratic convergence can be reduced to sub-quadratic, and some sort of regularization or adaptive step control would be necessary to ensure convergence.

Additionally, some memory effects in the system influence the convergence due to its fractional nature. The fractional derivative plays a role in the iterative process by means of a nonlocal structure that could be expressed by:

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} F(s, x(s))ds \qquad (8)$$

This is an integral representation, which complicates the convergence analysis with respect to classical ones, because it considers the influence of the previous states on the current approximations (Ramrao & Ingle, n.d.).

Another factor which affects the stability of the method is the selection of the initial approximation. A good initial guess is such that the iterations stay in a region of convergence, which may be defined as follows:

$$|x_0 - x^*| < \delta \qquad (9)$$

In singular systems, this region is often smaller, requiring careful selection of starting values.

Additionally, the study highlights that incorporating regularization techniques improves convergence stability. A modified iteration scheme can be expressed as:

$$x_{n+1} = x_n - \frac{G(x_n)}{G'(x_n) + \lambda} \qquad (10)$$

The overall discussion confirms that the Newton-Raphson method is still a powerful method but it needs to be adapted so it can be applied to singular fractional systems. Combination of fractional calculus, singularity handling and iterative methods gives a complete framework to solve complex nonlinear problem(Kannaujiya et al., 2025).

6. Conclusion

This work provides a complete convergence study of Newton–Raphson method for nonlinear fractional differential equation with singularities. The results show that the classical method is applicable to fractional systems as long as proper modifications are made to address the singular behaviour and nonlocal dynamics.

The research shows that the convergence is significantly affected by the type of singularities, the fractional order of the system as well as the initial approximation. Unlike smooth regions, convergence is not guaranteed in the vicinity of singularities and stabilization methods like regularization and adaptive iteration schemes may be required.

The use of fractional calculus in the context of Newton-Raphson framework gives a better insight into the effect of memory on iterative convergence. Numerical results show the proposed modifications enhance the accuracy, stability, and computational efficiency, suitable for complex real-world applications.

Overall, this paper connects the classical numerical method with modern fractional systems and provides a solid method to solve nonlinear problems with singularities. The results help in both theoretical analysis and work with advanced mathematical models.

7. Recommendations

Firstly, adaptive Newton type methods that adjust the iteration parameters according to the degree of singularity and fractional order is recommended for further research. The adaptive schemes can also be applied to further improve convergence stability and efficiency for highly nonlinear systems.

Secondly, it is important to explore hybrid numerical methods, mixing the Newton-Raphson method with other numerical methods like wavelet or finite element methods. This integration can enhance the precision of the solutions and can handle singularities in a complicated domain more accurately.

Thirdly, it is crucial to extend the existing framework to multi-dimensional and system-level fractional differential equations. The method is often applicable to real-world applications involving coupled systems, and studying convergence in these kinds of applications will extend the use of the method.

Last but not least, it is imperative to promote the application of the proposed methods in engineering and scientific applications. These techniques could be applied to real-world problems, like fluid dynamics, material science or financial modeling, to test their efficacy and offer insights to further enhance them.

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