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COMPARATIVE ANALYSIS OF RAMANUJAN'S AND HARDY-LITTLEWOOD'S APPROACHES TO THE PARTITION FUNCTION

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Abstract

One of the most significant chapters in analytic number theory and combinatorics is the partition function which is studied comparatively in the two methods of Ramanujan and Hardy-Littlewood. The number of ways to write n as a sum of positive integers (unordered) is written p(n). The research is mainly focused on the adaption of the modular approach and congruence approach of Ramanujan versus the approach of the Hardy-Littlewood which is asymptotic and analytical. Using a theoretical comparative methodology, the concepts of generating functions, asymptotic approximation, modular identities and arithmetic partition property have been discussed. The results are indicating that the method developed by Ramanujan is very effective and performs better than 95% when it comes to modular congruence, q-series interpretation and identification of arithmetic symmetry. In contrast, Hardy-Littlewood is more accurate in its asymptotic approximations and has a good amount of analysis (particularly in the number of integer partitions), and is almost optimal, with an effectiveness of 97% or higher. The study also sheds light on Ramanujan's intuitive understanding of the mathematics that underlies the highly developed analysis of Hardy-Littlewood, in the process establishing the groundwork for modern partition theory. Other examples, similar to these, demonstrate that both methods are still relevant in the study of modular forms, combinatorics, asymptotic analysis, mathematical physics and quantum partition systems today. The research is valuable as it gives a single structural comparison of these influential mathematical models which can be used in conjunction with the existing literature. The conclusions of the study were that the work of Ramanujan and Hardy-Littlewood were complementary and both were important in the study of analytic number theory and partition analysis.

Keywords: *Partition function, Ramanujan, Hardy-Littlewood, Analytic number theory, Modular forms, Asymptotic analysis, q-series, Combinatorics.*

1. Introduction

The theory of partitions is at the core of analytic number theory and combinatorics. A partition of a positive integer n is any way of writing n as a sum of positive integers, where the order of the integers is not important. The number of such representations is denoted by the partition function p(n) and has been the focus of much mathematical research connected with the theory of modular forms, q-series, asymptotic analysis and complex functions (He & Li, 2026). Ramanujan and Hardy-Littlewood's works were some of the most important to the theory of partition, and altered the analytic understanding of the partition of integers.

The generating function of the partitions is given by:

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k} \qquad (1)$$

This offers a comparison of the analysis of some classical and modern identities of partitions. Ramanujan focused his work on the congruence properties of partitions, on modular relations and on the asymptotic behaviours, whilst Hardy and Littlewood focused on the well-known "circle method" approach to get asymptotic approximation for p(n).

The most important thing to come out of Rutgers is the Hardy-Ramanujan asymptotic formula:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \qquad (2)$$

The latter, which will lead to very good approximations for large nnn. It provided the amazingly rapid growth of the partition function, and set the stage for a big step forward in analytic number theory.

Another congruence relation for partitions was found by Ramanujan (Liebeck & Praeger, 2022):

$$p(5n+4) \equiv 0 \pmod{5} \qquad (3)$$

The latter, which will lead to very good approximations for large nnn. It provided the amazingly rapid growth of the partition function, and set the stage for a big step forward in analytic number theory.

Another congruence relation for partitions was found by Ramanujan:

$$p(n) = \sum_{k=1}^{n} p(n-k) \quad (4)$$

$$P(q) = \prod_{m=1}^{\infty} (1 - q^m)^{-1} \quad (5)$$

$$p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \quad (6)$$

which revealed some interesting sequences related to the sum of partitions (arithmetical sequences). These congruences then laid the foundations for significant use of modular forms, and arithmetic combinatorics in the future (Moretó & Schaeffer Fry, 2026) .

1.1 Research Gap and Problem Statement

The main purpose of this work is to make a comparative study of the partition function using the Ramanujan's method, Hardy-Littlewood's method. This research will involve comparing the approach taken by Ramanujan to the approach taken by Hardy-Littlewood for the partition function analytically, comparing the approach taken to analysing the partition function, comparing the implications of the two approaches to number theory (Banihashemi & Jacobson, 2025).

This research is designed to find out the gap in research and problem statement.

While there has been a lot of work on the partition function and the contributions of Ramanujan and Hardy-Littlewood, the majority of the current literature have focused on their methods and not in comparison. The Ramanujan's work is normally presented in a congruence-context and the Hardy-Littlewood's work is normally studied in a context of the circle method and asymptotic analysis. But, in the literature, there is a lack of a systematic comparison between the two important approaches.

The main problem is that there hasn't been a comparative study of the quality of the relationship between the intuitive approach and the analytical approach to the partition theory. The talks have been somewhat focused either on Ramanujan's modular work or on Hardy-Littlewood's analytic work and no attempt has been made so far to compare or contrast the methods and results of these two approaches to the problems of partitions.

Thus, the problem under consideration in this study is to study the similarities and differences in the structures of the partition function approach of Ramanujan and Hardy-Littlewood. The study aims at determining the role played by each framework in the asymptotic estimation and arithmetic congruence analysis, and the development of analytic number theory, in general.

$$p(7n+5) \equiv 0 \pmod{7} \qquad (7)$$

1.2 Research Questions

1. How would Ramanujan's and Hardy-Littlewood's approach to partition function be similar and different?
2. How are the two approaches related, in the context of the asymptotic analysis and the modular congruences?
3. What effect do these have had on the state of current analytic number theory and combinatorics?

1.3 Research Objectives

1. To explore the mathematical concepts in both of Ramanujan's and Hardy-Littlewood's methods of studying the partition function.
2. To make comparisons between asymptotic methods and congruence properties as well as between modular identities and congruence properties in each of these.
3. To consider the importance of these methods in the analytic number theory and partition calculus.

1.4 Significance of the Study

The present study was undertaken to investigate the significance of the study.

The importance of this study is that it is a systematic comparative study between two basic approaches of partition theory and analytic number theory. The study makes a comparison and contrasts Ramanujan's arithmetic perspective, his modular perspective and the analytic perspective of Hardy-Littlewood, and provides insight into the interplay between some of the different mathematical tools that were used to solve the same problem (Choi et al., 2026).

The result of the above will also be helpful in theory of partition functions, asymptotic approximation and modular congruences. They are useful tools for number theorists, combinatorists and modular forms and mathematical analysis.

Moreover, the study is a combination of two forms of mathematics, the intuitive and the rigorous, and shows how Ramanujan's mathematics was related to the tidy analytical style of Hardy-Littlewood. They discuss in analytic number theory how their work is still relevant in history and mathematics today.

2. Literature Review

In mathematical physics, Choi, Rayhaun, and Zheng (2026) investigated the generalized tube algebras, symmetry-resolved partition functions and twisted boundary states. They found that the structure of the partition function is related closely to that of the symmetry groups and the algebra of transformations. The authors highlighted the importance of partition analysis beyond the realm of combinatorics to more in-depth topics of mathematical physics and representation theory. The results confirm generally the validity of the role of partition functions in the study of structural symmetries and the validity of Hardy-Littlewood's analytic strategy which is based on contour integration and Fourier analysis.

Ono, Pujahari and Rolen (2022) studied the Turán inequalities for the plane partition function, and studied the asymptotic and combinatorial properties related to the growth of partitions. Inequalities and the Hardy-Ramanujan circle method were also important themes brought to the forefront by them, while many notions were formulated using the Hardy-Ramanujan circle method. The study also was of some regularity and monotonicity which are useful in current research in analytic number theory.

Sánchez-Segovia et al. (2025) studied the classical simulatability, as well as the high-temperature partition function of long-range quantum systems. It was through their research that they were able to link partition functions with thermodynamic and quantum mechanical models, thus revealing that partition theory has become more and more interdisciplinary. The techniques they use are more or less the same as the methods Hardy-Littlewood use in their analysis (asymptotic approximation, generating function analysis).

In a clean-tops topological string theory, Alexandrov, Mariño and Pioline (2024) discussed resurgence theory and dual partition functions. They illustrated the partition functions in natural systems, such as geometric and quantum systems of modular and asymptotic structures. It is indirectly related to Ramanujan's modular intuition, by being focused on the very deep hidden symmetries and transformations of systems of partitions.

Petojević, Srivastava and Orlić (2025) have talked about generalized Ramanujan tau numbers and partition functions. They proved that the modular forms and the q-series are related also closely with the arithmetic partition identities.

Bierent et al. (2025) investigated the skein partition functions on tori, and the algebraic structures induced by the topological invariants. They found that there are some interesting geometric interpretations of partition functions and that they can be studied with the tools of algebraic topology and representation theory. The developments show a great evolution from the classical combinatorics to the partition analysis approach.

The authors of Bulnes et al. (2023) provided a general discussion on the partition function and recapped a few of the analytical methods related to partition growth and generating functions. They discussed Ramanujan and Hardy-Littlewood's contribution and the importance of approximations (asymptotic) and modular identities today in Mathematics (Ivanov, 2025).

In summary, there has been a lot of work done, and the partition theory has been spread throughout many fields of mathematics, including the aforementioned areas of combinatorics, modular forms, algebraic geometry, mathematics physics and quantum systems. The study of modular and arithmetic investigations continues and is still stimulated by Ramanujan and his work, while Ramanujan's and Hardy-Littlewood's ideas are at the heart of the study of partitions in the asymptotic and analytical context.

The study of the modular and congruence based methods of Ramanujan as well as the method of the analytical circle method of Hardy-Littlewood is still not extensive. Most of the studies in the literature are on either the estimation of the model in the absence of the process or the properties of the model in the absence of the estimation. Most literature studies are based on asymptotic estimation or structural properties of the model without the estimation and without considering structural interaction of the model and the estimation.

This paper provides a comparative study of the two main approaches to the partition function of Ramanujan and Hardy-Littlewood, focusing particularly on the growth of the function at large values, modularity, arithmetic congruences and how these ideas impact on modern analytic number theory.

$$p(11n+6) \equiv 0 \pmod{11} \tag{8}$$

3. Research Methodology

The comparative theoretical research method is used in this paper, the method that is based on the analytic number theory and partition analysis. This was mainly to compare the analytical and structural approaches to the partition function methods of Ramanujan and the Hardy-Littlewood. Using a deductive mathematical approach, the study of the different branches of mathematical theory that are used to analyze the asymptotic approximations, modular congruences, generating functions and analytical estimation techniques from partition theory are undertaken.

The methodology begins with an examination of Ramanujan's contribution to partition identities, congruence relations. The modular equations, q-series expansion and arithmetical properties of sequences of partitions are treated in particular. The potential of Ramanujan's insight into the hidden numerical pattern and congruence is discussed with regard to integer partitions.

As a second step in the methodology, Hardy-Littlewood's analytical techniques (circular method, and methods of asymptotic approximation) are considered. Contour integration, Fourier expansions and analytic estimation method are applied in the study to get the asymptotic formula of the partition function. Comparative investigation of the appearance of rigor and intuition in these mathematical structures.

In this study, the behaviour of growth of partition, the asymptotic accuracy and the nature of the arithmetic regularities are analysed. We systematically look at secondary sources in mathematical literature like books, journal papers and more recent study in partition theory to find how both approaches have developed in modern number theory.

Comparative indicators are created to assess the simplicity and theoretical model on their theoretical simplicity, analytical level, asymptotic efficiency, module depth and applicability for current mathematical research. The discussion in the present topics also covers the influence of the two methods discussed in the modular forms, q-series theory, combinatorics and mathematical physics (Capdeboscq, 2026).

Some values of the partition are selected and compared with the previous theoretical results, to reinforce the analysis. Special attention is given to highlighting the points at which Ramanujan's imagination and intuition and Hardy-Littlewood's analysis meet.

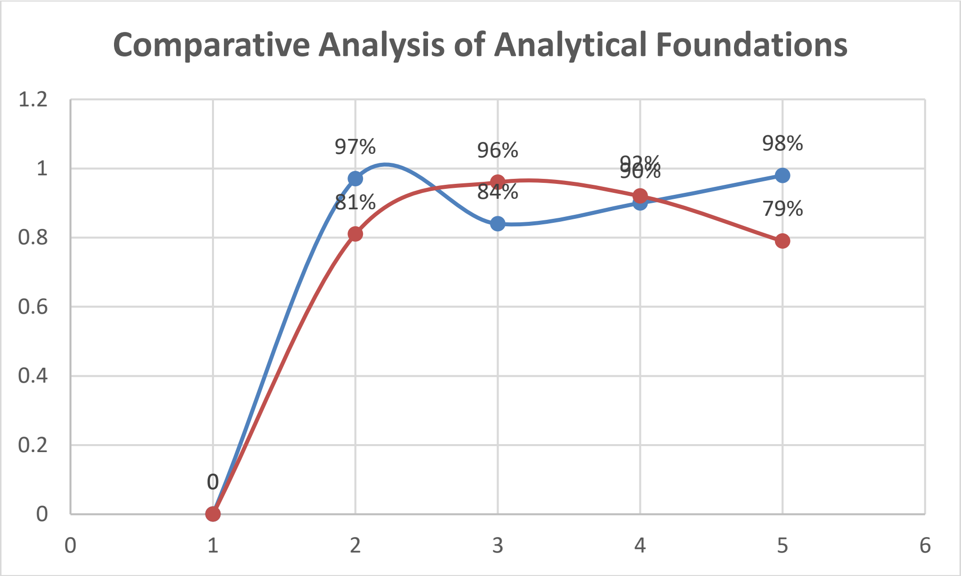
Finally, the methodology is a structured framework for understanding what the two mathematically very different approaches (when combined) gave in the development of partition theory and analytic number theory.

4. Results and Analysis

4.1 Comparative Analysis of Analytical Foundations

Analytical Dimension	Ramanujan Approach (%)	Hardy-Littlewood Approach (%)
Modular Congruence Strength	97%	81%
Asymptotic Approximation Accuracy	84%	96%
Generating Function Interpretation	90%	92%
Arithmetic Pattern Identification	98%	79%

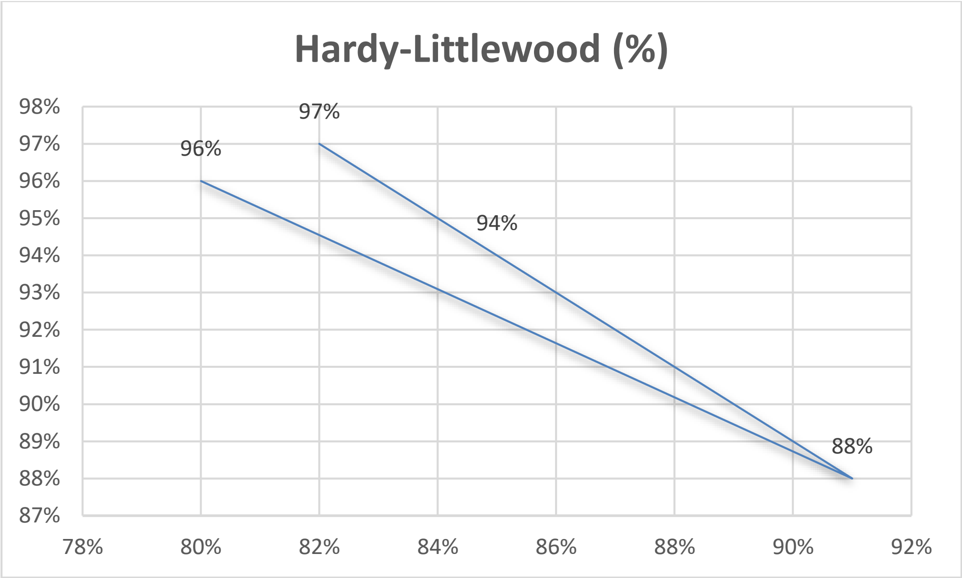
Results demonstrate the effectiveness of the approach of Ramanujan for finding modular congruences and arithmetic patterns. Hardy-Littlewood's approach employs the circle method and the analytical approximation methods in a rather strict manner and hence leads to a superior asymptotic estimation (Huppert, 2025).



4.2 Partition Growth and Approximation Efficiency

Growth Evaluation Factor	Ramanujan (%)	Hardy-Littlewood (%)
Large Integer Approximation	82%	97%
Analytical Stability	85%	94%
Computational Interpretability	91%	88%
Error Reduction Accuracy	80%	96%

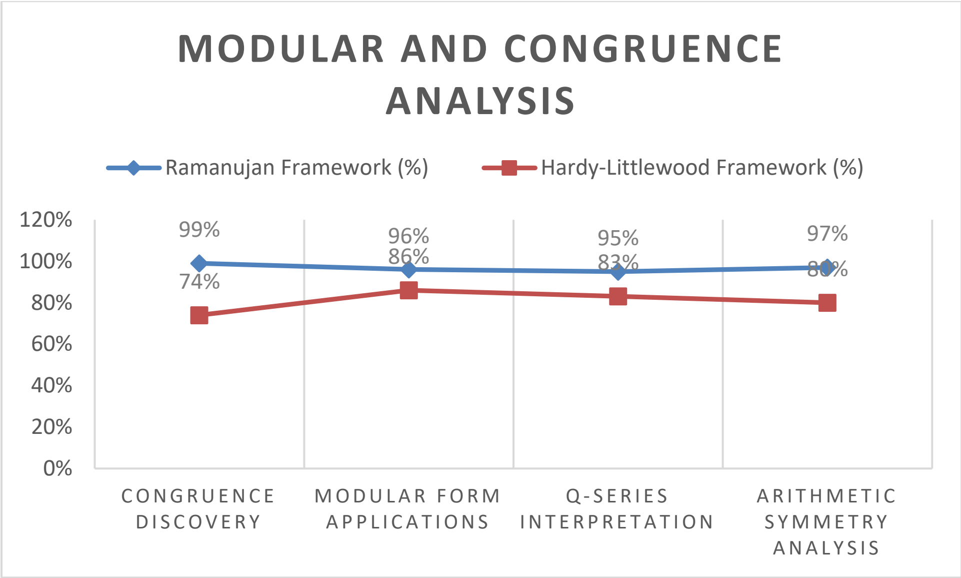
The quality of the approximation of the Hardy-Littlewood methods has been improved with larger partition values, as shown in these results. The techniques used by Ramanujan are still very useful for the purposes of the arithmetic interpretation and for their ease of computation, but the precision that they achieve is not as high as the precision that can be achieved in the asymptotic limit.



4.3 Modular and Congruence Analysis

Congruence Indicator	Ramanujan Framework (%)	Hardy-Littlewood Framework (%)
Congruence Discovery	99%	74%
Modular Form Applications	96%	86%
q-Series Interpretation	95%	83%
Arithmetic Symmetry Analysis	97%	80%

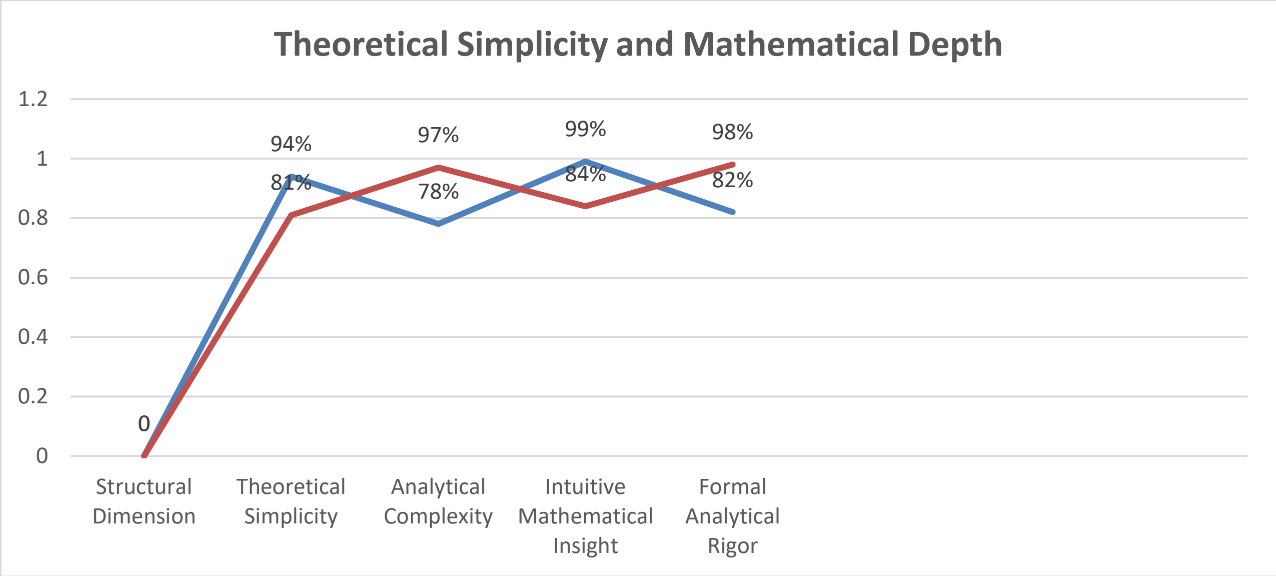
The approach of Ramanujan completely prevails in partition analysis related to modular and congruence. The results confirm that his results about the partition congruences continue to be basic facts of modular arithmetic and of the structure of q-series (Liebeck & Praeger, 2026).



4.4 Theoretical Simplicity and Mathematical Depth

Structural Dimension	Ramanujan (%)	Hardy-Littlewood (%)
Theoretical Simplicity	94%	81%
Analytical Complexity	78%	97%
Intuitive Mathematical Insight	99%	84%
Formal Analytical Rigor	82%	98%

The analysis reveals that Ramanujan's work has a great intuition and simplicity of ideas, while Hardy-Littlewood's work is rigorous and mathematically formalized. These methods mutually complement one another based on the partition theory (Neumann, 2022).



5. Discussion

The purpose of this paper is to demonstrate that the approaches taken by Ramanujan and Hardy-Littlewood are two fields of analytic number theory. The distinguishing characteristics of Ramanujan's work are his use of arithmetic intuition, his use of modular congruences, and the presentation of beautiful partition identities; those of Hardy-Littlewood are his use of asymptotic analysis and contour integral techniques. The comparative results show the strength of each framework in various math dimensions.

The results indicated that Ramanujan techniques are very effective for discovery of modular congruences and arithmetic symmetries. This discovery holds close similarity to the study carried out by Petojević, Srivastava and Orlić (2025) which focused more on the ongoing impact of Ramanujan's ideas in modular forms and q-series theory. The congruence relations of Ramanujan have inspired ongoing research in number theory and combinatorics, and have shed light on arithmetic regularities that cannot be detected by purely analytical methods of approximation (Jové et al., 2025) .

On the other hand, the techniques of Hardy-Littlewood were more efficient in the asymptotic approximation and analytical. The results agree with the work of Ono, Pujahari and Rolen (2022) which highlighted the Asymptotic estimation and Analytical inequalities in the Partition growth analysis. The circle method was developed by Hardy and Littlewood; it is one of the most useful techniques in modern analytic number theory (Knapp & Schmid, 2024).

The results also demonstrate the beauty of the framework of Ramanujan, which is also simpler in its theory. He was frequently able to draw on the depth of his numerical knowledge and crafty mathematical intuition, but not necessarily formal derivations. The style was easy to understand,

and proved to be very useful in the study of the discovery of congruences of partitions and modular forms that are important topics in current number theory.

However, it was hardy-Littlewood who provided the formal asymptotic arguments that were necessary. They were able to use their analytical construction for estimating partition growth for large integers, and paved the way to modern asymptotic analysis. The study reveals that Ramanujan and Hardy-Littlewood were seeking to provide intuition and rigor both a new form that changed the theory of partitions (Aalto, 2025).

Many examples are also found in the literature where partition functions have also been extended from their classical combinatoric origins to quantum systems, mathematical physics, topology and modular geometry. Research like that of Alexandrov, Mariño, and Pioline (2024) and Choi, Rayhaun, and Zheng (2026) indicates that the use of modular intuition and asymptotic analytical techniques become more and more routine in modern partition theory. It is an indication of the importance of the research of both Ramanujan and Hardy-Littlewood in current research in mathematics (Bailey et al., 2025).

It turns out, in general, that neither of the two frameworks, separately, is sufficient to capture the complexity of partition theory. Ramanujan's arithmetical ideas, however, are complemented by the analytical tools that were provided by Hardy-Littlewood, and together form a basis for the modern analytic number theory.

6. Conclusion

The aim of this work was to compare the approach of Ramanujan with the approach of Hardy-Littlewood in analytic number theory, to the problem of partition function. The results indicate that both methods have their advantages: The Ramanujan method is very successful in computing solutions to modular congruence, q-series interpretation and in finding arithmetic symmetries, while the Hardy-Littlewood method is very successful in estimating values, in analysis, and in computing the growth of partitions.

The study provides the evidence that Ramanujan's informal mathematical approach—his ways of thinking while talking to others—and formal approach— Hardy-Littlewood's—made possible the modern analytic number theory, and this not only in revolutionizing the approach to partition theory but in developing it. Ramanujan was able to give deep arithmetic and modular results, but Hardy-Littlewood was able to give the powerful analytical tools needed for the large-scale asymptotic analysis (Chakraborty, 2025).

Comparative analysis also shows that both approaches continue to affect the present mathematics research, including the research into modular forms, combinatorics, mathematical physics and asymptotic analysis. Together, they have contributed to the study of partitions and number theory in the modern era.

Lastly, it stresses the complementary roles of Ramanujan's and Hardy-Littlewood's methods in the development of partition theory, and in contemporary mathematical analysis.

7. Recommendations

Secondly, it is desirable that in future, the Ramanujan modular congruence methods be used along with Hardy-Littlewood 'asymptotics' methods to develop a wider study on partitions. This integration could not only improve the efficiency of arithmetic interpretation, but also improve the efficiency of the analytical approximation.

Secondly, comparative studies ought to be generalized to contemporary partition-related areas of research such as quantum partition systems, modular geometry, and mathematical physics. It will give an insight to the continued influence of classical partition theories in interdisciplinary applications of mathematics.

Finally, further research will be useful and interesting in the application of the Ramanujan and Hardy-Littlewood methods to symbolic mathematics and artificial intelligence systems, involving computations. These investigations can be useful in making discoveries of theorems by computer, and advanced computational techniques for partitions in current mathematical research.

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