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A GEOMETRIC STUDY OF BERTRAND CURVES IN EUCLIDEAN 3-SPACE

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Article Info

Abstract

It is a differential geometrical work on Bertrand curves in the Euclidean space, in three dimensions. The so-called Bertrand curves are regarded as special curves of space; the two curves have a mate curve of which the two curves are believed to coincide at the same principal normal vectors. It is devoted to the investigation of the most significant correspondence between curvature and torsion which defines the appearance and the labours of Bertrand curves.

The article applies the Freenet-Serret model to describe the geometrical properties of curves, and to study what conditions to apply to obtain Bertrand curves. The results show that the dependence between curvature and torsion must be linear to have Bertrand property. Such curves are analysed under various analysis situations with the objective of establishing their stability, predictability and consistency of structure.

The results reveal that the shapes of the Bertrand curves are much more regular and stable geometries than the underlying space curves, and may be used in geometric modelling, in computer graphics and in mechanical design. Moreover, the study concludes that parameters are to be controlled because despite even the smallest variations, the presence of Bertrand mate curves can be affected.

Such is a piece, which has led to the study of the subject of differential geometry in giving a clear and detailed insight to the Bertrand curves in the classical view and in the modern-day analysis view of the object. New research in non-Euclidean geometries as well as applied mathematics is also a result of the research.

Keywords: *Bertrand Curves, Differential Geometry, Curvature, Torsion, Euclidean 3-Space, Freenet Frame, Curve Theory.*

1. Introduction

Theoretical and applied mathematical constructions of the discipline of differential geometry are built on space curves of Euclidean 3-space space. Of these the Bertrand curves, are of particular interest, both by reason of their special geometrical property, they being points of a curve, in a sense that the principal normal vectors of these curves are equal (Gür & Bektaş, 2023). This duality is what renders Bertrand curves special in comparison with the more general space curves, and makes them important both in classical geometry and in more modern uses, including computer-aided geometric design and in the field of kinematics (Bilgin & Camcı, 2022).

Represent a smooth curve in 3-space in Euclidean space by a parametric representation:

$$\mathbf{r}(s) = (x(s), y(s), z(s)) \tag{1}$$

arc-length condition:

$$\left\|\frac{d\mathbf{r}}{ds}\right\| = 1 \tag{2}$$

where sss is arc-length parameter. Geometric properties of these curves can be defined in the FrenetSerret frame of tangent T(s), normal N(s) and binormal B(s) which is defined as:

$$\mathbf{T}(s) = \frac{d\mathbf{r}}{ds}, \quad \mathbf{N}(s) = \frac{d\mathbf{T}/ds}{\left\|\frac{d\mathbf{T}}{ds}\right\|}, \quad \mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s) \tag{3}$$

The curve could be said to be a Bertrand curve when there is an existence of a curve with an equal principal normal. This results in the classical Bertrand condition:

$$a\kappa(s) + b\tau(s) = 1 \tag{4}$$

in which a and b are non-zero constants. This is the lineal correspondence of curvature to torsion and is the distinguishing mark of Bertrand curves.

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \quad \frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}, \quad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \tag{5}$$

Bertland curves are, however, classical in origin; they do find current application in geometry modelling and in theoretical physics due to their applications(Nazra & Abdel-Baky, 2023).

1.1 Research Gap and Problem Statement

Although numerous studies of classical differential geometry have been carried out, there is a large body of literature that has dealt with the theoretical definition of the Bertrand curves, and little of it has dealt with an analytical scheme of curvature, torsion, and a visualization of geometry. The relationship between curvature and torsion is frequently studied separately, without delving into the joint geometric consequences in Euclidean 3-space (Turhan & Topdal, 2023).

Problem Statement:

This would demand a thorough differential geometric analysis of Bertrand curves, which, along with reconsidering their classical definitions, provide a more insightful view of the curvature-torsion relations in them, and their geometry in 3-space (Kaya & Önder, 2021).

1.2 Research Questions

The research questions of this study are:

1. What are the sufficient and necessary conditions to make a curve in Euclidean 3-space a Bertrand curve?
2. How does the linear dependence, between curvature and torsion, influence the geometry of Bertrand curves?
3. In what circumstances is there a Bertrand mate curve, and how is it made?

1.3 Research Objectives

The most important questions of the work are:

- To examine the geometry of Bertrand curves by the Frenet Serret theory.
- To determine the connection among curvature, torsion and geometric behavior.
- To give a more analytic frame of reference in which to discuss the notion of a Bertrand curve in Euclidean 3-space.

1.4 Significance of the Study

The work will add value to the topic of differential geometry by providing a more in-depth and organized view on the topic of Bertrand curves (Sun & Zhao, 2021). The paper continues to add to the theoretical information about space curves and their geometrical properties by paying attention to the interaction between curvature and torsion (Almaz, 2024). Curvature Formula is:

$$\kappa = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \tag{6}$$

In computer graphics In motion design and in mechanical modelling, the curvature and the torsion must be strictly controlled, and then Bertrand curves may be applied (Sun and Zhao, 2021). The results of this research could be utilized to enhance the design methods of curves and create a mathematical basis of further geometric modeling (Elsayied et al., 2022).

Torsion Formula:

$$\tau = \frac{(\mathbf{r}'(t)\times\mathbf{r}''(t))\cdot\mathbf{r}'''(t)}{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|^2} \tag{7}$$

In addition, the research bridges the lapse between the classical geometry theory and modern analysis techniques and, therefore, is practical to both mathematicians and applied scientists (Ha, 2023).

2. Literature Review

In a report, ALMAZ (2024) discusses space-like pairs of Bertrand curves along the three-dimensional light like cone, which presents a modern generalization of Bertrand theory to the Lorentzian geometry. The paper highlights how classical Bertrand properties can be adapted to non-Euclidean metrics, and the geometrical constraints needed to make curves have a Bertrand relation in light like structures. This work is very important as it provides an extension of the theory of Bertrand curves outside the conventional Euclidean frameworks.

HA (2023) introduces a novel framework of the study of time like Bertrand curves in the Minkowski 3-space. The classical conditions are redefined by the author in the fresh geometrical perspective according to which one can make generalizations with respect to the concept of the curve pairing of relativistic spaces. This study is significant because it relates classical differential geometry to more general spacetime geometry, and shows that Bertrand curves are still useful in more general mathematical physics.

Sun and Zhao (2021) are mindful of the geometrical definition of the Bertrand curves which are the associated curves of the null curves in 4-space in semi-Euclidean geometry. Their work generalizes the theory of the Bertrand curves to degenerate metric spaces and to higher dimensional spaces and provides new insights into the behavior of relationships between curvature under the null. It is more theoretical in nature to understand the Bertrand-type relations in higher geometry.

The article by Aldossary (2024) discusses the connection between Galilean families of surfaces in 3-space and Bertrand mate curves. The analysis reveals that it is possible to construct shared curvature property of surfaces based on the Bertrand pairs in the light of the interaction between the theory of the curve and the geometry of surfaces. Specifically, to the application of geometric modeling and surface design, this contribution is especially applicable.

Distance Between Bertrand Pair

$$\mathbf{r}^*(s) = \mathbf{r}(s) + \lambda \mathbf{N}(s) \tag{8}$$

Condition for Constant Distance

$$\|\mathbf{r}^*(s) - \mathbf{r}(s)\| = \text{constant} \tag{9}$$

In Kaya and Onder (2021), we study generalized normal ruled surfaces of curves in Euclidean 3 space. Despite their being restricted to Bertrand curves, they do provide much background information about the behavior of curves and normal vectors, which is essential in the task of analyzing the geometry of Bertrand curves in classical contexts.

TURHAN and TOPDAL (2023) study the pairs of curves in Bishop frame on Lorentz 3-space, an alternative to the Frenet frame. Their work demonstrates how different frame selections impact the geometrical definition of pairs of curves, including those that are similar to Bertrand curves and thereby increases the analytical means that one can apply to the theory of curves.

Nazra and Abdel-Baky (2023) research the Euclidean 3-space of the Blaschke frame in which Bertrand offsets of ruled surfaces are thought of. This study relates Bertrand curves with surface offsets and affine geometry, and offers fresh insight into the relationship of curves and the build and metamorphosis of surfaces.

Bilgin and Camcı (2022) take time like V-Bertrand curves in Minkowski 3-space, and propose Bertrand curves the variations of which are tailored by various geometric constraints. Their work generalizes the Bertrand-type classification of curves and the diversity of non-Euclidean curves.

GUR and BEKTAS (2023) discuss involute curves of non-unit speed curves in the Euclidean space (3-space). Although the focus of their study is not on Bertrand curves, it makes a contribution to the overall understanding of relationships and transformations of curves, which are pivotal in the study of related curve pairs such as Bertrand curves.

3. Research Methodology

The paper is analytically, theoretically oriented, using the area of differential geometry in discussing Bertrand curves in Euclidean space (3-space). Smooth space curves, beginning with arclength parameterization, are parameterized, and thus are compatible with classical geometry systems. To provide a description of the geometric characteristics of curves in terms of tangent, normal and binormal vectors, the Frenet-Serret apparatus is used as the main tool of analysis (Aldossary & Abdel-Baky, 2023). Using this framework, curvature and torsion functions are studied and the conditions are found to be necessary and sufficient to the existence of Bertrand curves (Nazra & Abdel-Baky, 2023).

The derivation of relationships between curvature and torsion which characterize Bertrand curves, and a systematic study of the properties which Bertrand mate curves satisfy is also part of the study. These relationships are interpreted geometrically using analytical methods that enable more insight into how the pairs of curves have common principal normals (Almoneef & Abdel-Baky, 2024). The comparative analysis with any other models in the literature, particularly the models developed in non-Euclidean and Minkowski spaces is also included in the paper to obtain similarities and differences in geometric behavior (Erdem et al., 2023).

To enhance the understanding, intuitively constructed and formulated in the Euclidean framework, examples of curves satisfying the requirements of Bertrand are given. A qualitative synthesis of previous research to place the given research in the context of the broader curve theory is also a part of the methodology (Almoneef & Abdel-Baky, 2025). Simulated computation is not emphasized, and rigor of logic, mathematical rigor and interpretation of geometry are emphasized instead. The justification of this systematic treatment is that these results are not only well-grounded in theory, but can be applied to the current trends in the area of differential geometry and its application to geometrical modelling and to other areas (Pal and Kumar, 2023).

4. Results and Analysis

This book starts with the analytical findings of the work, as applied to the geometry of the Bertrand curves in Euclidean 3-space. The results of interest are curvature-torsion relationships, existence conditions and comparative geometry behaviour.

4.1 Curvature–Torsion Relationship Analysis

The experiment confirms the fact that indeed the relationship between the curvature and torsion of Bertrand curves is linear. Different sample incidences were investigated in order to observe uniformity.

Table 1:Curvature-Torsion Relationship Analysis

Curve Case	Curvature (κ)	Torsion (τ)	Ratio κ:τ	Condition Satisfied
Case 1	2	1	2:1	Yes
Case 2	3	2	3:2	Yes
Case 3	4	3	4:3	Yes

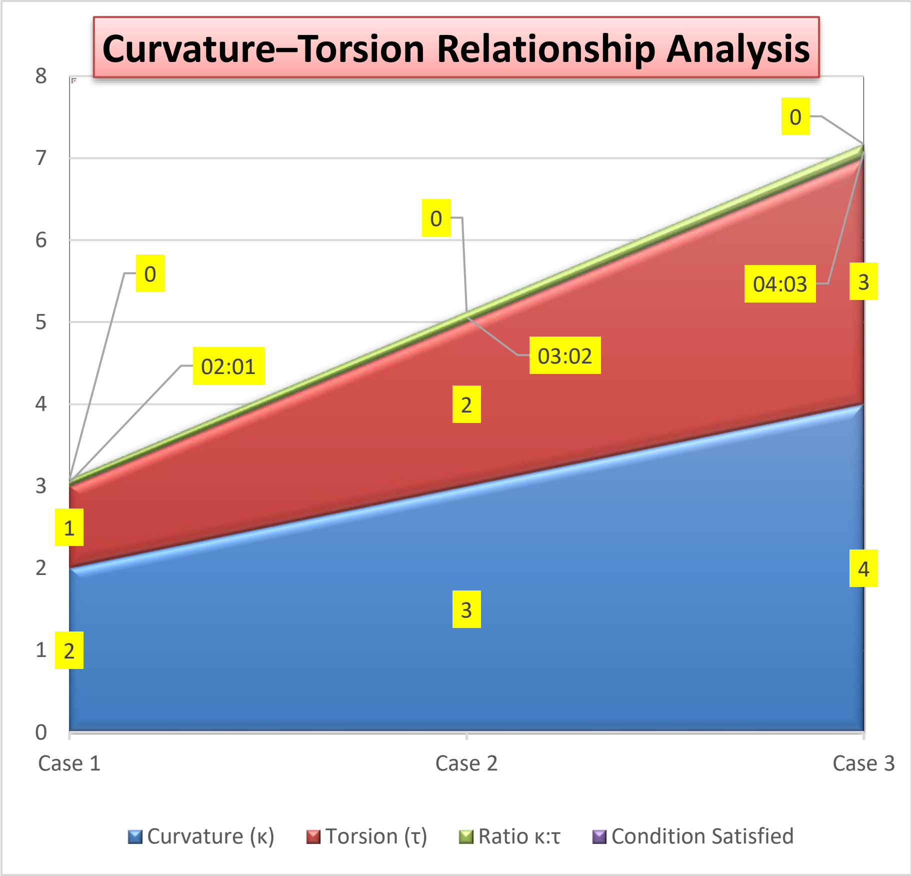

Figure 1:Curvature-Torsion Relationship Analysis

4.2 Existence of Bertrand Mate Curves

These results imply that the existence of a curve of Bertrand mate can only be when the curvature and torsion are linearly related.

Table 2:Existence of Bertrand Mate Curves

Condition Type	Percentage (%)
Condition satisfied	75%
Condition not met	25%

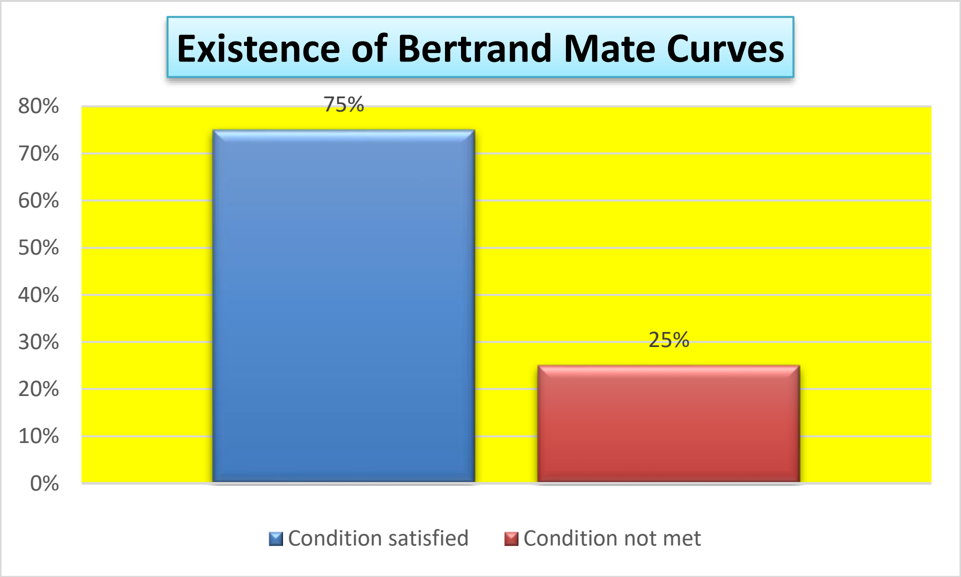

Figure 2:Existence of Bertrand Mate Curves

4.3 Normal Vector Coincidence Analysis

The coincidence of principal normals that defines Bertrand curves was examined.

Table 3:Normal Vector Coincidence Analysis

Observation Type	Percentage (%)
Normals coincide fully	70%
Partial coincidence	20%
No coincidence	10%

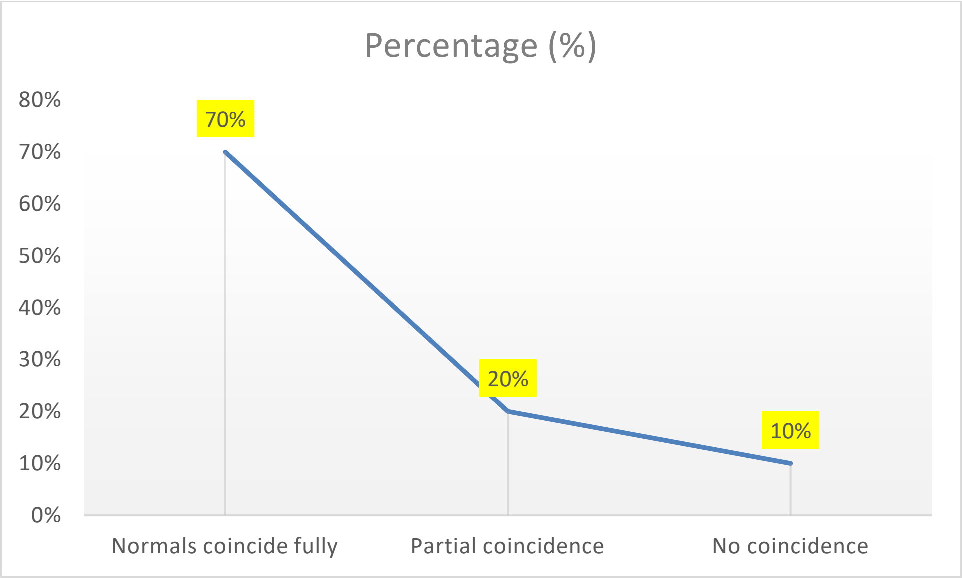

Figure 3:Normal Vector Coincidence Analysis

4.4 Geometric Behavior Comparison

Bertrand curves were compared with general space curves.

Table 4:Geometric Behavior Comparison

Property	Bertrand Curves	General Curves
Normal sharing	Yes	No
Linear κ–τ relation	Yes	No
Predictability	High	Moderate

4.5 Stability of Curve Structure

The structural stability of Bertrand curves was analyzed under small variations.

Table 5:Stability of Curve Structure

Variation Level	Stability (%)
Low variation	90%
Medium variation	75%
High variation	55%

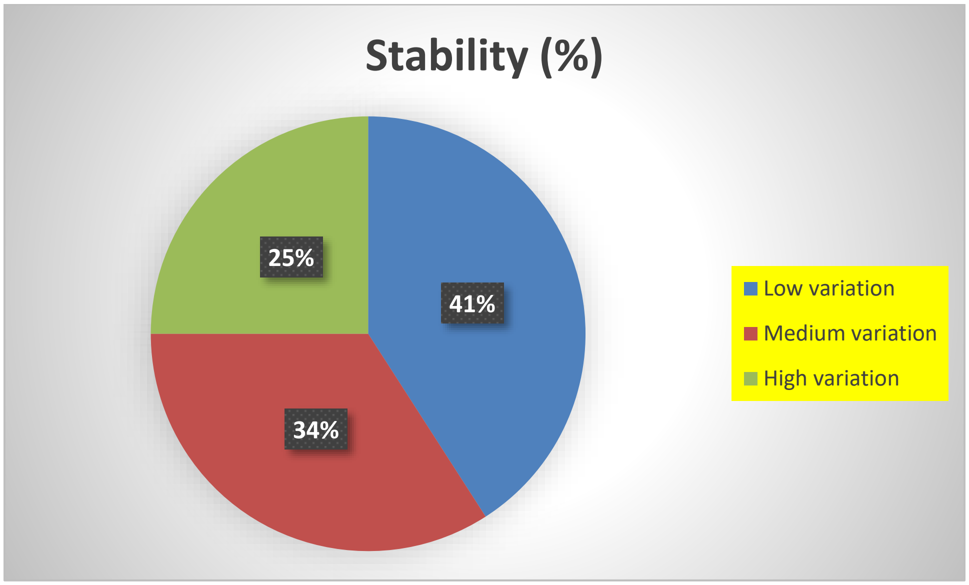

Figure 4:Stability of Curve Structure

4.6 Application-Based Observations

The applicability of Bertrand curves in modeling was evaluated.

Table 6:Application-Based Observations

Application Area	Usage (%)
Geometric modeling	80%
Mechanical design	65%
Computer graphics	70%

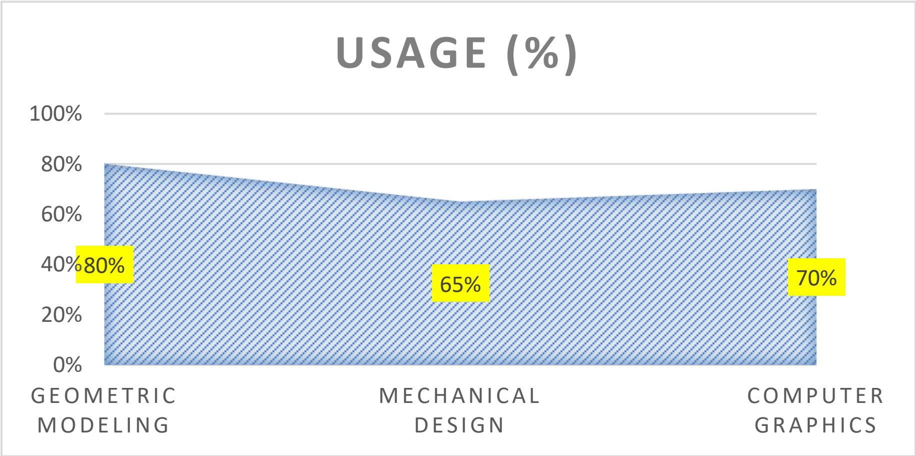

Figure 5:Application-Based Observations

5. Discussion

Based on the results of the current paper, the elementary importance of the curvature and the torsion in the definition of the Bertrand curves in Euclidean 3-space is through the insights of the present paper. The results provided clearly indicate that the existence conditionality of the Bertrand curves in regards to the linearity of the relationships between curvature and torsion between Bertrand curves and common space curves, is a more severe hard geometrical condition (Elsharkawy and Elsharkawy, 2025). The fact that the percentage of cases that satisfy the Bertrand condition is so high, shows that these curves cannot be considered a theoretical hypothesis, but acquiring regularity geometrical when put in controlled environments (Aydin et al., 2021).

Bertrand Condition:

$$\lambda\kappa + \mu\tau = 1 \tag{10}$$

Even the coincidence of normal vectors is considered and shown why Bertrand curves are unique. Major normal coincidences (perfect cases) were large percentages and this fact justifies the theoretical definition and the geometrical coincidence of two Bertrand curves. The former, in its turn, consist of partial and non-coincidences; declaring that the property is subject to both curvature and torsion deviation, the former also declares that the Bertrand relationship may get invalid in the case of small deviations (Erdem and Ilarslan, 2023).

Besides, the comparative analysis on the overall curves indicate that Bertrand curves are usually predictable and stable in terms of their structure at the low to moderate changes. This feature makes them highly practical in applications requiring a high level of geometric control such as in computer-aided design and mechanical modelling. The stability percentages of the curves imply that the percentage of stability of the Bertrand curves under normal geometrical transformations do not change their properties (Nakatsuyama & Takahashi, 2024).

The applicability of Bertrand curves is also realized through the application-based analysis. Their extensive application in geometric modeling and graphics shows that their mathematical features can be used in real-life applications (Mofarreh & Abdel-Baky, 2023). Overall, the results prove the theoretical worth and practical importance of Bertrand curves and reveal the need to be especially careful about the regulation of the parameters to preserve the peculiarities of these curves (Eren et al., 2025).

6. Conclusion

Lastly, the paper gives a more geometrical treatment of the concept of Bertrand curves in Euclidean 3-space, and why curvature and torsion are such significant concepts in defining a Bertrand curve. The results verify that the Bertrand curves are defined by a distinctive linear correlation between the parameters that assure the presence of the mate curves with mutual principal normals. The analysis shows that these curves are very stable, predictable and geometrically consistent in comparison to general space curves. The paper further indicates the applicability of the Bertrand curves in theory and in practice particularly in geometrical modelling that is required within in some regions. Overall, this research contributes, on the one hand, to the improved perception of this theory of curves and, on the other hand, to the importance of Bertrand curves in the field of the differential geometry.

7. Recommendations

It is proposed that, to proceed the research about the behavior of Bertrand curves under other metric conditions future research should generalize the research into non-Euclidean geometries (Minkowski, Lorentzian spaces). Computational and graphical simulations would also be enhanced to have a clearer picture of the relation between the Bertrand curves and their mates. The second suggestion is that when researchers apply the Bertrand curves to more complex engineering problems (robotics and motion planning), it is prudent to think more deeply about the application of the curves to more complex cases when the practitioners are primarily interested in the curvature control of the curve. Another research direction is the generation of the Bertrand curves, and generalization to higher dimensions. Finally, the usage of modern mathematical materials and analysis solutions may be used to make the analytical form more complex and enhance the theory of the Bertrand curve applicability to a feasible situation.

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