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A COMPARATIVE ANALYSIS OF CHEBFUN AND MATLAB'S BVP4C METHOD FOR SOLVING THE FISHER-KPP EQUATION

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Article Info

Abstract

This study investigates numerical solutions to the Fisher-KPP equation, a nonlinear ordinary differential equation (ODE) used in modeling population dynamics and reaction-diffusion processes. We compare two solvers: MATLAB's bvp4c, a widely-used method for boundary value problems, and Chebfun, which utilizes Chebyshev polynomial approximations for potentially improved accuracy. The performance of both solvers is assessed in terms of accuracy, computational efficiency, and effectiveness for solving the Fisher-KPP equation. The findings highlight the strengths and limitations of each method, offering guidance on selecting the most appropriate solver for nonlinear problems in computational science and engineering.

Keywords:
Fisher-KPP equation, Chebfun, MATLAB, bvp4c, nonlinear ODE, numerical methods

Introduction

Spectral methods emerged as a powerful alternative in the 1980s, leveraging global approximations based on orthogonal polynomials, such as Chebyshev or Legendre polynomials [1]. These methods provide exponential convergence for sufficiently smooth solutions, making them ideal for problems like the Fisher-KPP equation, where the solutions are smooth and well-behaved [2]. Nevertheless, spectral methods require significant expertise to implement and are less effective for problems involving discontinuities or sharp transitions.

The Chebyshev spectral collocation method and Chebfun have been extensively employed in solving various types of differential equations due to their accuracy and efficiency. These methods have demonstrated significant success in tackling ordinary differential equations (ODEs), partial differential equations (PDEs), delay differential equations (DDEs), and stochastic differential equations (SDEs). For instance, Trefethen et al. (2004) applied the Chebyshev collocation method to linear and periodic delay differential equations, showcasing its reliability in handling such problems [3]. Additionally, Wang et al. (2015) explored the application of the Chebyshev spectral collocation method to stochastic delay differential equations, highlighting its potential for solving equations with inherent randomness [4]. Moreover, Wu et al. (2024) presented a comprehensive framework for applying the Chebyshev spectral collocation method to functional and delay differential equations, further extending its applicability [5]. These studies collectively underscore the versatility and effectiveness of these methods in addressing complex mathematical models, making them essential tools in modern computational mathematics.

MATLAB's bvp4c, a robust built-in solver, has gained popularity for solving boundary value problems (BVPs), including nonlinear and stiff equations [6]. Based on finite difference collocation methods, bvp4c is both reliable and easy to use. However, its accuracy is limited compared to spectral methods, particularly for problems requiring high precision or global representations of the solution [7].

Chebfun, a MATLAB-based software package, revolutionized numerical methods by simplifying the implementation of spectral techniques [8]. At its core, Chebfun represents functions as piecewise polynomial interpolants over Chebyshev nodes, achieving spectral accuracy with minimal user input. It abstracts the complexity of traditional spectral methods, enabling researchers to solve differential equations with the simplicity of a scripting interface [9].

The foundation of Chebfun lies in Chebyshev interpolation, which minimizes numerical errors by distributing interpolation nodes according to a cosine spacing. This approach ensures stability and accuracy even for highly nonlinear equations like the Fisher-KPP equation [10]. Chebfun also leverages the Clenshaw-Curtis quadrature for numerical integration, which is faster and more precise than standard integration techniques. For nonlinear systems, Chebfun employs Newton-Raphson iterations, achieving rapid convergence for well-posed problems.

1.2 Background of the Fisher -KPP equation

The Fisher-KPP equation is a pivotal tool in analyzing reaction-diffusion systems, introduced independently by Ronald A. Fisher in 1937 [11] and Kolmogorov, Petrovskii, and Piskunov (KPP) in the same year [12]. Initially proposed to describe the propagation of advantageous genetic traits within a population, the Fisher-KPP equation has since become a universal framework for modeling the interaction between diffusion and nonlinear reactions. Its applications span fields such as mathematical biology, population genetics, ecology, epidemiology, and combustion theory [13]. The equation’s ability to describe traveling wave solutions, which represent phenomena like the spread of invasive species or flame fronts, has cemented its significance in scientific modeling [14].

Mathematically, the general time-dependent Fisher-KPP equation is expressed as

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ru(1-u),$$

where u(x,t) represents the population density or concentration of a substance at position and time, D is the diffusion coefficient, and r is the intrinsic growth rate [15]. For steady-state solutions, the time derivative vanishes, simplifying the equation to:

$$u'' + ru(1 - u) = 0,$$

With boundary conditions:

$$u(0) = 0, \qquad u(1) = 1.$$

The term models logistic growth, introducing nonlinearity that makes analytical solutions impractical except in special cases [16]. Consequently, numerical techniques are frequently used to solve the Fisher-KPP equation [17].

Kolmogorov et al. explored the equation from a probabilistic perspective, laying the foundation for broader applications of reaction-diffusion systems. Over the years, the Fisher-KPP equation has been applied to describe processes as diverse tumor growth, chemical reaction kinetics, and the invasion of ecosystems by non-native species [18]. Beyond genetics, the Fisher-KPP equation has been applied extensively in epidemiology to model the spatial spread of infectious diseases. For instance, Hosono et al. (1995) demonstrated how mathematical models could predict the rate at which diseases like influenza or cholera spread geographically [19]. In ecology, Shigesada et al. (1995) applied the equation to model the spread of invasive species, offering significant insights into how external influences can disrupt ecological equilibria [20]. In combustion theory, Barenblatt et al. (1985) used the equation to provide predictive insights into reaction front stability and dynamics, particularly in describing flame propagation in reactive gases [21]. According to H. L. Smith in [22], the Fisher-KPP equation's universality is found in its capacity to represent situations in which the interaction of growth/reaction and diffusion leads to wave-like behaviors. The traveling wave solutions of the Fisher-KPP equation, which emerge under certain boundary conditions, are one of its most remarkable features. These solutions describe fronts moving with constant velocity, where the population transitions from one stable state to another. The speed and shape of these waves are determined by the equation's parameters, offering a quantitative framework for understanding a wide range of phenomena [23]. Analytical methods provide only limited insights into these solutions, particularly when boundary conditions or nonlinearities deviate from the simplest cases. As such, numerical methods play a critical role in advancing our understanding of this equation [24].

Strikwerda (2004) highlighted the foundational role of finite difference methods (FDM) in the numerical analysis of the Fisher-KPP equation. These methods approximate derivatives by computing differences between values on a discrete grid, offering simplicity and ease of implementation in various applications. [25]. However, LeVeque (2007) observed that finite difference methods (FDM) face limitations, including numerical instabilities and the need for fine grids to effectively capture sharp gradients or boundary layers. [26]. Finite element methods (FEM) offer greater flexibility in handling irregular domains and provide more accurate approximations of solutions, particularly for systems with complex boundary conditions [27]. However, FEM often incurs higher computational costs due to its reliance on locally defined basis functions.

This paper has been organized as follows: The second section discusses the Fisher-KPP equation's numerical solution. The technique for determining the approximate solution will be covered in the third part. Lastly, the fourth portion contained the paper's concluding remarks.

Here we are using the standard form of the Fisher-KPP Equation to obtain its solution

$$u'' + u(1-u) = 0 \tag{1}$$

With Boundary Conditions:

$$u(-4) - 1 = 0, \qquad \mathrm{u}(4) = 0. \tag{2}$$

2. Numerical Solution of the Fisher-KPP Equation

First, we solve the problem using the bvp4c function in MATLAB, as shown in Fig. 1.

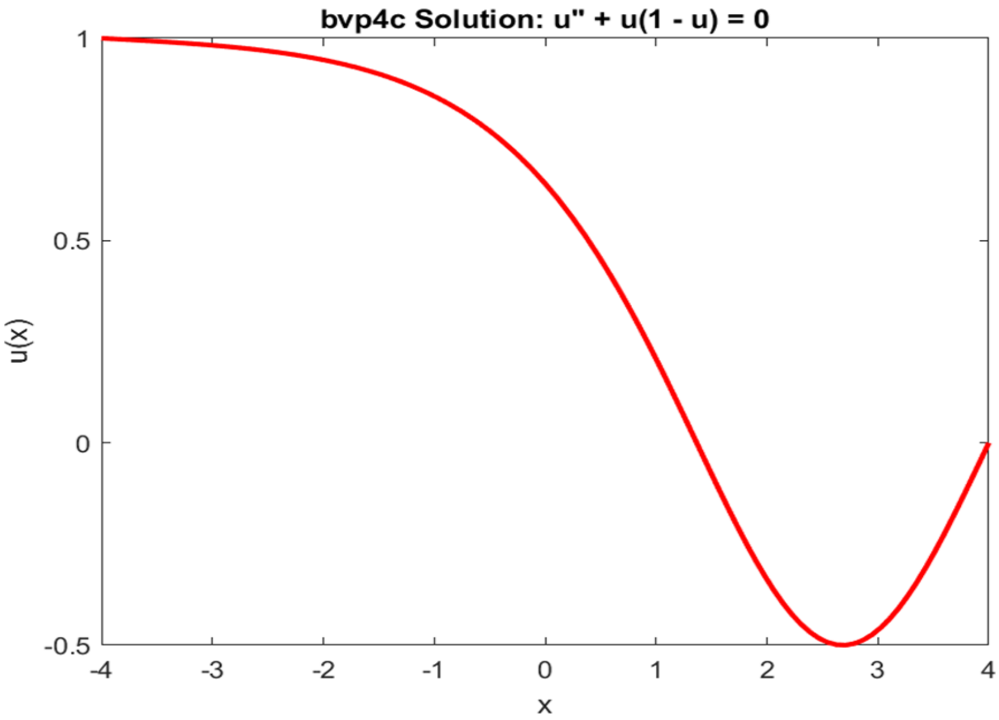

Fig.1: Numerical Solution of the Fisher-KPP Equation using bvp4c.

3. Approximate Solution of the Fisher-KPP Equation

Chebfun is an open-source software system primarily designed for computational tasks involving Chebyshev polynomials and their derivatives. This MATLAB-based software suite provides a collection of algorithms that are easily accessible online. While it traditionally works with Chebyshev polynomials, users have the flexibility to modify the domain for specific needs. The core idea behind Chebfun is that smooth functions can be effectively represented by polynomial interpolation at Chebyshev points, offering an efficient computational approach. One of its key features, chebop, is used for solving differential equations through the spectral collocation method. This tool integrates the domain, operators, and boundary conditions, all tailored to the specific problem being addressed. For the solution of Eq.1, chebfun and chebop are used.

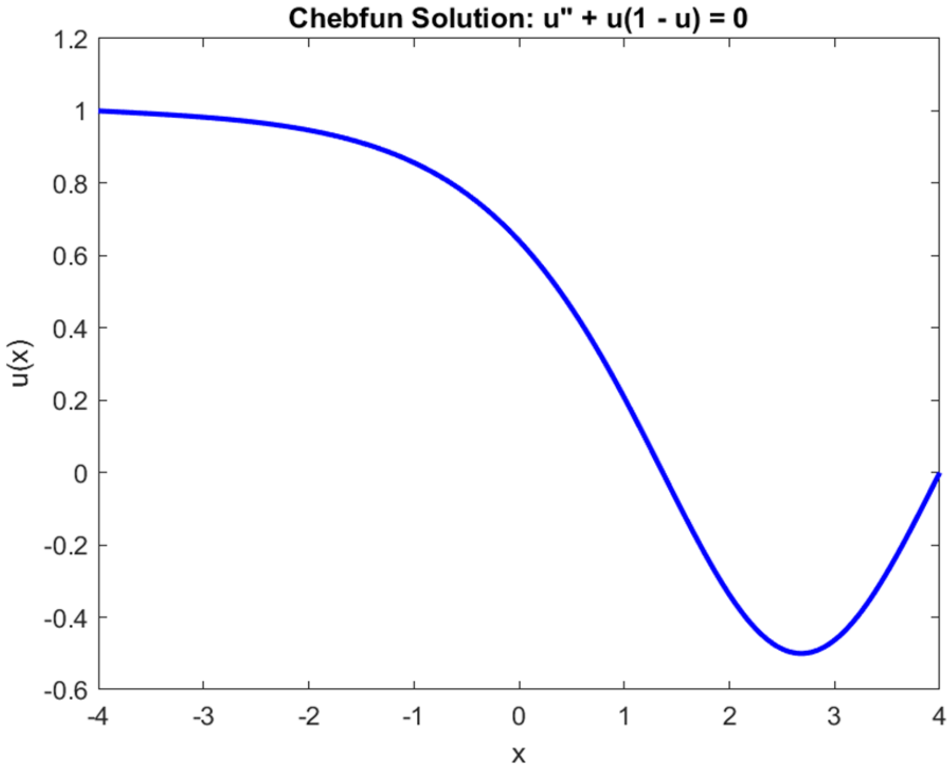

Fig.2. Approximate Solution of the Fisher KPP Equation using Chebfun.

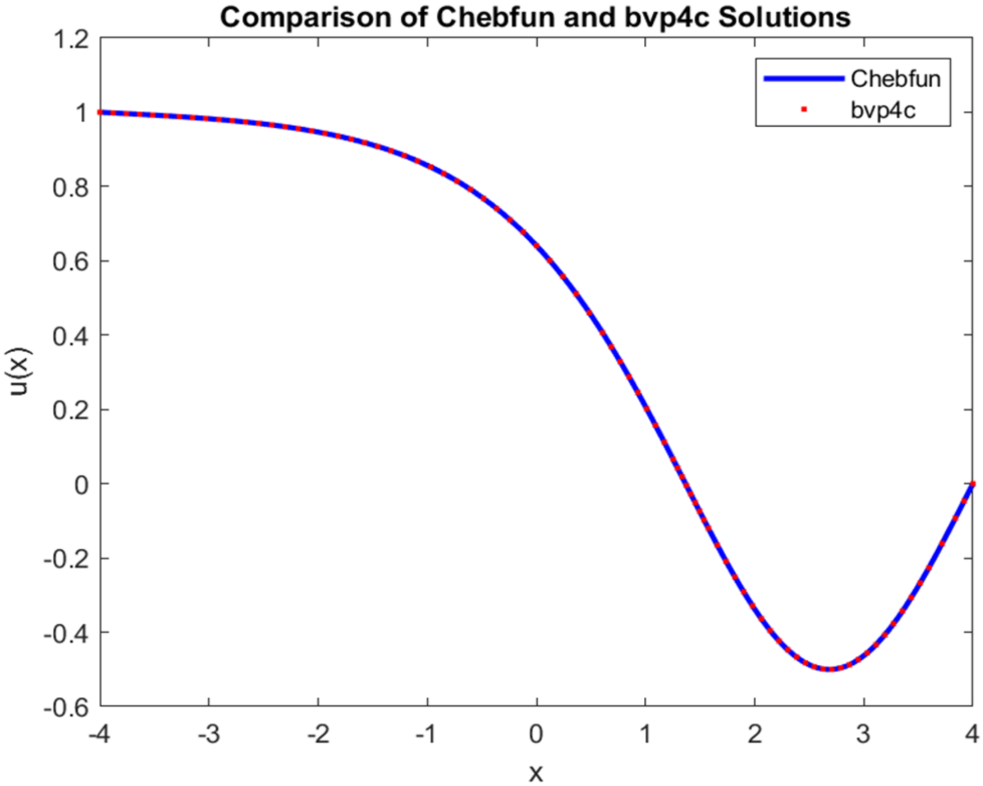

Fig.3. Comparison of Chebfun and bvp4c.

In fig.3 the results obtained using spectral methods (by chebfun) are compared with results by bvp4c to validate the accuracy of the spectral collocation method.

Table 1: Approximation of u(x)for Chebyshev spectral method and bvp4c

x	$Chebfun\ u(x)$	$bvp4c\ u(x)$
-4.00	1.00000000	1.00000000
-3.43	0.99103552	0.99103553
-2.86	0.97909745	0.97909747
-2.29	0.96031310	0.96031313
-1.71	0.92880281	0.92880287
-1.14	0.87528763	0.87528775
-0.57	0.78566969	0.78566983
0.00	0.64094824	0.64094838
0.57	0.42230572	0.42230559
1.14	0.12798825	0.12798810
1.71	-0.19659988	-0.19660014
2.29	-0.44243116	-0.44243141
2.86	-0.48884088	-0.48884108
3.43	-0.30956216	-0.30956227
4.00	-0.00000000	0.00000000

The results displayed in Table 1 demonstrate that there is a high degree of agreement between the two sets of data. This comparison demonstrates the results consistency and stability, indicating that the two methods yield results which are similar.

Conclusion

In this paper, the nonlinear Fisher-KPP equation has been approximated using the spectral method. To accomplish this objective, we utilize Chebfun, a MATLAB-based tool designed for solving numerical problems through spectral methods, specifically employing Chebyshev polynomials.

On the other hand, bvp4c was used as well to determine the numerical solution to the required problem. A close agreement between two outcomes is indicated by the comparison of the results in Table (1).

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