OPEN ACCESS

Kashf Journal of Multidisciplinary Research
Vol: 02 - Issue 2 (2025)
P-ISSN: 3007-1992 E-ISSN: 3007-200X
https://kjmr.com.pk

SOME GEOMETRIC PROPERTIES OF RABOTNOV FUNCTION

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***DOI:** https://doi.org/10.71146/kjmr266*

Article Info

Abstract

This article focuses on the geometric properties of normalized Rabotnov function. We use constructive tactics to establish the conditions for close-to-convexity and find conditions under which the normalized Rabotnov function is star-like and prestarlike. We also apply the starlike function $\vartheta/(1-\vartheta^2)$ to establish the conditions for close-to-convexity.

Keywords:
Rabotnov functions, starlike function, close to the convex function, prestarlike function; strategies for optimism.

Introduction

The most well-known and extensively researched class of analytic functions f, denoted by λ, have the form

$$\lambda(t) = t + \sum_{n=0}^{\infty} a_n t^n$$

A function is referred to as convex if it maps Ʊ into a convex domain and starlike if it maps Ʊ onto a domain that is starlike regarding to origin. The class of all starlike as well as convex univalent functions belongs to Ʊ is represented by $\c{S}^*$ and $\hat{C}$, respectively. The definitions of $\c{S}^*$ and $\hat{C}$'s generalizations, represented by $\c{S}^*(\sigma)$ (starlike) and $\hat{C}(\sigma)$ (convex) of order $\sigma[0,1]$ respectively, are

$$\c{S}^*(\sigma) = \left\{ f : \mathbb{R}\left(\frac{\vartheta f'(\vartheta)}{f(\vartheta)}\right) > \sigma, \qquad \vartheta \in \text{Ʊ} \right\},$$

and

$$\hat{C}(\sigma) = \left\{ f : \mathbb{R}\left(1 + \frac{\vartheta f''(\vartheta)}{f'(\vartheta)}\right) > \sigma,\ \vartheta \in \text{Ʊ} \right\}.$$

For the order of σ, the close to convex function is defined as

$$\Bbbk(\sigma) = \left\{ f : \mathbb{R}\left(\frac{\vartheta f'(\vartheta)}{g(\vartheta)}\right) > \sigma, \qquad \vartheta \in \text{Ʊ}, g \in \c{S}^*(0) \right\}.$$

The convolution of functions (Hadamard product) is defined by

$$(f * g)(\vartheta) = \vartheta + \sum_{v=2}^{\infty} c_v c_v \vartheta^v \qquad (\vartheta \in \text{Ʊ}).$$

Using this convolution concept, Ruscheweyh [1] developed a class $\mathfrak{R}_{\mathfrak{F}}$, it includes the following prestarlike functions of order $\mathfrak{F}$,

Let $f \in$ Å. Then, $f \in \mathfrak{R}_{\mathfrak{F}}$ if and only if

$$\begin{cases} \mathbb{R}\frac{f(\vartheta)}{\vartheta} > 0, \vartheta \in \text{Ʊ}, & \text{for } \mathfrak{F} = 1 \\ \frac{\vartheta}{(1-\vartheta)^{2(1-\mathfrak{F})}} * f(\vartheta) \in \c{S}^*(\mathfrak{F}),\ \vartheta \in \text{Ʊ} & \text{for } 0 \le \mathfrak{F} < 1. \end{cases}$$

When we put $\mathfrak{F} = \frac{1}{2}$ then

$\hat{C} = \mathfrak{R}_0$ and $\c{S}^*\left(\frac{1}{2}\right) = \mathfrak{R}_{1/2}$

The class $\mathfrak{R}[\alpha, \mathfrak{F}]$ was created by generalizing the class $\mathfrak{R}_{\mathfrak{F}}$ by Sheil-Small et al. [2].

A function $f \in \mathfrak{R}[\alpha, \mathfrak{F}]$ if $f * \c{S}_\alpha \in \c{S}^*(\mathfrak{F})$, where $\c{S}_\alpha = \frac{\vartheta}{(1-\vartheta)^{2-2\alpha}}$, $0 \le \alpha < 1$.

It is observed that $\mathfrak{R}[\mathfrak{F}, \mathfrak{F}] = \mathfrak{R}_{\mathfrak{F}}$.

$$f(\bar{\vartheta}) = \overline{f(\vartheta)}.$$

An analytical continuation to the entire complex plane is the aforementioned extension formula [3]. The Harmonic Functions Minimum Principal (MP) states that unless a harmonic function u is constant, it can't have minimum or maximum at an interior point [4].

The **Robertsnov functions**, also known as **Rabotnov functions**, are special functions used in viscoelasticity theory, particularly in the modeling of hereditary materials. These functions are instrumental in describing the time-dependent stress-strain relationships in materials that exhibit both elastic and viscous behavior. Yuri Nikolaevich Rabotnov (1914–1985) was a prominent Russian scientist renowned for his contributions to mechanics, particularly in the field of viscoelasticity. In 1948, he introduced a fractional operator to model the behavior of materials with memory effects, a concept central to viscoelastic theory [5]. This operator, now known as the Rabotnov fractional-exponential function. Rabotnov's work laid the foundation for the application of fractional calculus in modeling the hereditary properties of materials, allowing for a more accurate description of stress-strain relationships over time. Despite being aware of the connection between his fractional operator and fractional derivatives, Rabotnov preferred to work with integral equation methods. His pioneering efforts have significantly influenced contemporary studies in viscoelasticity and continue to be a cornerstone in the field.

The Rabotnov function is defined by

$$\mathfrak{K}_{\mu,\beta}(t) = t^{\mu} \sum_{k=0}^{\infty} \frac{\beta^k}{\Gamma((k+1)(1+\mu))} t^{k(1+\mu)}.$$

Here Γ represents the Gamma function. In practice, the Rabotnov function may be represented in various forms, often involving convolution integrals. For a material under stress $\sigma(t)$ and strain $\epsilon(t)$, the Rabotnov function $\mathfrak{K}(t)$ might be expressed as:

$$\sigma(t) = \int_0^t \mathfrak{K}(t-\mu) \frac{d_{\epsilon}(\mu)}{d\mu}\, d\mu,$$

Where $\mathfrak{K}(t)$ is a relaxation function characterizing the material response over time. The Rabotnov function $\mathfrak{K}_{\mu}(t)$ has integral representations involving exponential and power-law terms:

$$\mathfrak{K}_{\mu}(t) = \frac{1}{2\pi i}$$

Rabotnov [6] created the fractional exponential function, sometimes referred to as the Rabotnov fractional exponential function (RFEF), in 1948. The Rabotnov function [5] plays a crucial role in the mathematical modeling of scientific and technical problems [7]. Yang et al. [7] introduced a highly interesting non-integer-order derivative operator in the context of the FREF five years ago, in 2019. The authors approximated an arbitrary-order heat transfer equation using their operator [7]. For more details about the Rabotnov function, see [8] and [9].

For our main results, we use the following lemmas.

Lemma 1 [10] Consider the sequence $\{c_v\}_{v=1}^{\infty}$ of positive real numbers such that $c_1 = 1$. Let $c_1 \geq 8c_2$ and $(v-1)c_v - (1+v)c_{v+1} \geq 0, \forall_v \geq 2$. Then,

$$f(\sigma) = \sigma + \sum_{v=2}^{\infty} c_v \sigma^v \in K$$

with respect to starlike function $\frac{\sigma}{1-\sigma^2}$.

Lemma 2 If the function $f(\sigma) = \sum_{v=1}^{\infty} c_v \sigma^{v-1}$, where $c_1 = 1$ and $c_v \geq 0,\ \forall_v \geq 2$ is analytic in U, and if $\{c_v\}_{v=1}^{\infty}$ is a convex decreasing sequence, $i.e., c_{v+2} - 2c_{v+1} + c_v \geq 0$ and $c_v - c_{v+1} \geq 0, \forall_v \geq 1$, then $\mathbb{R}f(\sigma) > \frac{1}{2}, \forall_{\sigma} \in$ U.

2. Main Results:

Theorem#1 Suppose $a, b \geq 1$, and $\Gamma(a+b) \geq 8\Gamma(b)$, $2\Gamma(2a+b) \geq 3\Gamma(a+b)$ are satisfied. Then $\mathfrak{K}_{a,b} \in \mathcal{k}$ with respect to starlike function $\frac{\vartheta}{1-\vartheta^2}$.

Proof: Consider

$$\mathfrak{K}_{a,b}(\vartheta) = \vartheta + \sum_{v=2}^{\infty} c_v \vartheta^v$$

where

$$c_v = \frac{\beta^{v-1}\Gamma(1+\mu)}{\Gamma((1+\mu)v)} \quad \text{for } v \geq 1, and\ c_1 = 1$$

$\forall\ v \geq 2$, we have to show that c_v satisfies the conditions of lemma 1.
Clearly $a \geq 1$ as well as $b \geq 1$, the inequality
$\Gamma(a+b) \geq 8\Gamma(b)$
is satisfied.
Additionally,

$c_1 = 1$ and $c_1 >_{\approx} 8c_2$

$\Longrightarrow$

$$c_1 = \frac{\beta^{1-1}\Gamma(1+\mu)}{\Gamma\big((1+\mu).1\big)} = \frac{1.\Gamma(1+\mu)}{\Gamma(1+\mu)} = 1$$

Also

$$c_2 = \frac{\beta^{2-1}\Gamma(1+\mu)}{\Gamma((1+\mu).2)} = \frac{\beta\Gamma(1+\mu)}{\Gamma((1+\mu)2)}$$

$$c_1 >_{\approx} 8c_2$$

Again for $v \geq 2$, consider

$$(v-1)c_v - (v+1)c_{v+1} > 0$$

$$(v-1)\frac{\beta^{v-1}\Gamma(1+\mu)}{\Gamma((1+\mu)v)} - (v+1)\frac{\beta^{v}\Gamma(1+\mu)}{\Gamma\big((1+\mu)(v+1)\big)} > 0$$

$$(v-1)\frac{\beta^{v-1}\Gamma(1+\mu)}{\Gamma((1+\mu)v)} > (v+1)\frac{\beta^{v}\Gamma(1+\mu)}{\Gamma\big((1+\mu)(v+1)\big)}$$

$$(v-1)\frac{\beta^{-1}}{\Gamma((1+\mu)v)} > \frac{(v+1)}{\Gamma\big((1+\mu)(v+1)\big)}$$

$$\frac{(v-1)}{\beta\Gamma((1+\mu)v)} - \frac{(v+1)}{\Gamma\big((1+\mu)(v+1)\big)} > 0$$

$$(v-1)\Gamma\big((1+\mu)(v+1)\big) - (v+1)\beta\Gamma((1+\mu)v) > 0$$

Putting $v = 2$

$$\frac{\Gamma\big((1+\mu)3\big)}{\Gamma\big((1+\mu)2\big)} - 3\beta > 0$$

One can easily observe that the above expression for $a \geq 1,\ b \geq 1,$ is non-negative if $2\Gamma(2a+b) \geq 3\Gamma(a+b)$. It is evident that $\{c_v\}_{v=1}^{\infty}$ satisfies Lemma 1. This proved the result. ■

Theorem#2 Suppose that $a \geq 1, b \geq 1$, and $\Gamma(a+b) > \Gamma(b), \{2\Gamma(2a+b) + \Gamma(b)\}\Gamma(a+b) > 4\Gamma(b)\Gamma(2a+b)$, are satisfied. Then, $\mathbb{R}\left\{\frac{\mathfrak{K}_{a,b}(\vartheta)}{\vartheta}\right\} > \frac{1}{2}, for\ \vartheta \in$ Ʊ.

Proof. To find our results, we have to prove that the sequence

$$\{c_v\}_{v=1}^{\infty} = \left\{\frac{\beta^{v-1}\Gamma(1+\mu)}{\Gamma\big((1+\mu)v\big)}\right\}_{v=1}^{\infty}$$

is decreasing. Since

$\Gamma((1+\mu)v+1) > \Gamma((1+\mu)v)$ $\quad$ $(\forall\ v \geq 1, a \geq 1\ and\ b \geq 1)$

Therefore

$$\frac{\Gamma\big((1+\mu)v+1\big)}{\Gamma(b)} > \frac{\Gamma\big((1+\mu)v\big)}{\Gamma(b)}$$

$\Longrightarrow$

$$\frac{\Gamma(b)}{\Gamma\big((1+\mu)v\big)} > \frac{\Gamma(b)}{\Gamma\big((1+\mu)v+1\big)}$$

Now, we demonstrate that the sequence $\{c_v\}_{v=1}^{\infty}$ is decreasing and convex. For this we show that

$$c_v + c_{v+2} - 2c_{v+1} \geq 0$$

$$\frac{\beta^{v-1}\Gamma(1+\mu)}{\Gamma\big((1+\mu)v\big)} + \frac{\beta^{v+1}\Gamma(1+\mu)}{\Gamma\big((1+\mu)v+2\big)} - 2\frac{\beta^{v}\Gamma(1+\mu)}{\Gamma\big((1+\mu)v+1\big)} \geq 0$$

$$\Gamma(1+\mu)\left\{\frac{\beta^{v-1}}{\Gamma\big((1+\mu)v\big)} + \frac{\beta^{v+1}}{\Gamma\big((1+\mu)v+2\big)} - 2\frac{\beta^{v}}{\Gamma\big((1+\mu)v+1\big)}\right\} \geq 0$$

$$\Gamma\big((1+\mu)2\big)\Gamma\big((1+\mu)3\big) \geq 2\beta\Gamma(1+\mu)\Gamma\big((1+\mu)3\big) - \beta^2\Gamma(1+\mu)\Gamma\big((1+\mu)2\big)$$

$$\Gamma\big((1+\mu)2\big)\Gamma\big((1+\mu)3\big) \geq \beta\,\Gamma(1+\mu)\big[2\Gamma\big((1+\mu)3\big) - \beta\Gamma\big((1+\mu)2\big)\big]$$

$$\frac{1}{\beta\Gamma(1+\mu)} \geq \frac{2\Gamma\big((1+\mu)3\big) - \beta\Gamma\big((1+\mu)2\big)}{\Gamma\big((1+\mu)2\big)\Gamma\big((1+\mu)3\big)}$$

$$\frac{\Gamma(2+2\mu)\Gamma(3+3\mu)}{\beta\Gamma(1+\mu)} \geq 2\Gamma(3+3\mu) - \beta\Gamma(2+2\mu)$$

$$\frac{\Gamma(2+2\mu)\Gamma(3+3\mu)}{\beta\Gamma(1+\mu)} + \beta\Gamma(2+2\mu) \geq 2\Gamma(3+3\mu)$$

Which shows that the sequence $\{c_v\}_{v=1}^{\infty}$ is a convex decreasing. Now, from lemma 2 $\{c_v\}_{v=1}^{\infty}$ satisfy $\mathbb{R}(\sum_{v=1}^{\infty} c_v\,\vartheta^{v-1}) > \frac{1}{2}$, for all $\vartheta \in \mho$

Therefore,

$\mathcal{R}\left\{\frac{\mathfrak{K}_{a,b}(\vartheta)}{\vartheta}\right\} > \frac{1}{2}, for\ \vartheta \in \mho.$

Hence the result is proved. ■

Theorem#3 Suppose $a \geq 1$, $b \geq 1$ and $\Gamma(a+b) > 2\Gamma(b)$, $\{2\Gamma(2a+b) + 3\Gamma(b)\}\Gamma(a+b) > 8\Gamma(b)\Gamma(2a+b)$ are satisfied. Then the normalized Rabotnov function $\mathfrak{K}_{a,b} \in \S^*$.

Proof: To prove that $\mathfrak{K}_{a,b} \in \S^*$, we show that $\{vc_v\}$ and $\{vc_v - (v+1)c_{v+1}\}$ are non-increasing. Because $c_v \geq 0$ for $\mathfrak{K}_{a,b}(v)$ under the given conditions. So, let

$$\{vc_v - (v+1)c_{v+1}\} > 0$$

$$v\left[\frac{\beta^{v-1}\Gamma(1+\mu)}{\Gamma\big((1+\mu)v\big)}\right] - (v+1)\left[\frac{\beta^{v+1-1}\Gamma(1+\mu)}{\Gamma\big((1+\mu)(v+1)\big)}\right] > 0$$

$$v\frac{\beta^{v}\beta^{-1}\Gamma(1+\mu)}{\Gamma\big((1+\mu)v\big)} > (v+1)\frac{\beta^{v}\Gamma(1+\mu)}{\Gamma\big((1+\mu)(v+1)\big)}$$

$$\frac{v\Gamma\big((1+\mu)(v+1)\big) - \beta(v+1)\Gamma\big((1+\mu)v\big)}{\beta\Gamma\big((1+\mu)v\big)\Gamma\big((1+\mu)(v+1)\big)} > 0$$

$$\frac{\Gamma\big((1+\mu)2\big) - 2\beta\Gamma((1+\mu)}{\beta\Gamma((1+\mu)\Gamma\big((1+\mu)2\big)} > 0$$

$$\Gamma\big((1+\mu)2\big) > 2\beta\Gamma((1+\mu)$$

Now,

$$(v+2)c_{v+2} - 2(v+1)c_{v+1} + vc_v$$

$$\beta^{v}\Gamma(1+\mu)\left[\frac{(v+2)\beta}{\Gamma\big((1+\mu)(v+2)\big)}-\frac{2(v+1)}{\Gamma\big((1+\mu)(v+1)\big)}+\frac{v}{\beta\Gamma\big((1+\mu)v\big)}\right]$$

$$\beta\Gamma(1+\mu)\left[\frac{3\beta}{\Gamma\big((1+\mu)3\big)}-\frac{4}{\Gamma\big((1+\mu)2\big)}+\frac{1}{\beta\Gamma(1+\mu)}\right]$$

$$3\beta^{2}\Gamma(1+\mu)\Gamma\big((1+\mu)2\big)+\Gamma\big((1+\mu)2\big)\Gamma\big((1+\mu)3\big)>4\beta\Gamma(1+\mu)\Gamma\big((1+\mu)3\big)$$

$$3\beta^{2}\Gamma(1+\mu)+\Gamma\big((1+\mu)3\big)-\frac{4\beta\Gamma(1+\mu)\Gamma\big((1+\mu)3\big)}{\Gamma\big((1+\mu)2\big)}>0$$

Hence, the Normalized Rabotnov function $\mathfrak{K}_{a,b} \in \S^{*}$.

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