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ON NEARLY S-PERMUTABLE AND NEARLY HALL S-EMBEDDED SUBGROUPS: IMPLICATIONS FOR SUPERSOLVABILITY AND P-NILPOTENCY IN GROUPS

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Article Info

Abstract

A subgroup T of G is said to be nearly S-permutable if (p,|T|) =1 , for some prime p. Also, for every H≤G containing T the normalizer N_k (T) contain some Sylow p-subgroups of H. Furthermore, T is called nearly Hall S-embedded if for some normal subgroup N of G s.t TN (≤G) is a Hall and T∩N ≤ T ≤T_s (G), where T_s G is the largest nearly S-embedded of G contained in T and T is nearly S-permutable provided supplement B≤T such that T permutes with every Sylow P of B. Now we will show supersolvability and p-nil potency of some groups.

Keywords: *Nearly S-permutable, supersolvability, p-nilpotent, Hall subgroups*

MSC (2020): *20D10, 20D15, 20D20*

Introduction

Only finite groups will be discovered in this article. All the symbols and notations are standard. A finite group G and its sub groups denoted by H and T. Sylow subgroups of a finite group are denoted by $S_y l_G$. A permutable subgroup S is denoted as S H = H S, S H ≤ G. There are interesting results and generalizations related to S-permutable are shown in [3-6]. As subgroup T of G is called S-permutable if T Q = Q T ∀ Q $\mathcal{E}$ $S_y l_G$ (see [1]) S-permutable subgroups are also known as S-quasinormal and the subgroups consist of very interesting series of results.

For example; If a subgroup T is S-permutable than this subgroup is subnormal in G [1]. Also, from [2] the subgroup T H/ H_G is nilpotent. Now we will present the basic definitions of this paper.

Definition: 1.1

Suppose T(≤ G) and is known as nearly S-permutable ∀ P with (P, |T |) = 1 and ∀ T H ≤ G contains T. The symbol N_h(T) is normalizer containing $S_y l_p$(H).

Definition: 1.2

Subgroups T ≤ G is called nearly Hall S-embedded in G if for any N ≤ G satisfying TN a Hall subgroup of G ᵹ T ∩ N ≤ T ≤ T_s (G) where T_sG is the largest nearly S-permutable contained in T.

The group G with maximal Sylow subgroups, NH-embedded in G are supersolvable [8]

From nearly Hall S-embedded and nearly S-permutable subgroup it is clears that normal subgroups and Hall subgroups are nearly Hall S-embedded subgroups. On the other hand, converse does not hold. In this article, we will present some generalizations related to supersolvability and p-nilpotency and the following theorem will be proved.

Theorem: 1.3

If every maximal subgroup of every $S_y l$(G) is either nearly S-permutable or nearly Hall S-embedded, then G is supersolveable.

Theorem: 1.4

Suppose that for any finite group G and G_p $\mathcal{E}$ $S_y L_p$(G) for | G | $/p$ then p is the smallest number. Consider every maximal subgroup H ≤ G_p, either nearly S-permutable or nearly Hall S-embedded subgroup. Then G is $p - nilpotent$.

Theorem: 1.5

Suppose H ⊵ G (of a finite group) and $^{G}/_{H}$ be a supersolvable and every maximal subgroup every maximal subgroup T of S_yl(H) is either nearly S-permutable or nearly Hall S-embedded. Then G is supersolvable.

Preliminaries: 2

Here we will prove our basic results to generalize the idea of nearly Hall S-embedded subgroups of finite groups.

Lemma: 2.1 [7]

Suppose a NS-permutable Subgroup T ≤ G and M Δ G. Then

(1) TM is NS-permutable.
(2) If prime power order group T, then T ∩ M is NS-permutable in G.
(3) If prime power order so $^{\mathrm{TM}}/_{M}$ is NS-permutable in $^{G}/_{M}$ for any M Δ G.

Lemma: 2.2 [8]

|T| = q^n , Where q^n is a prime, so

T ≤ Q_q(G)

And T is NS-permutable in G.

Lemma: 2.3 [9]

Consider T ≤ G is S-permutable so T/T_G nilpotent, T_G is the core of T ≤ G.

Lemma: 2.4 [10]

If T ≤ G is S-permutable and T is a prime group with prime number q

So,

O^q(G) ≤ N_G(T)

Lemma: 2.5 [11]

Suppose T ≤ G, where T is nilpotent so,

(1) T ≤ G where T is S-permutable.
(2) S_yl(H) is S-quasinormal.

Lemma: 2.6 [12]

Consider Q to be a S_yl_q(G) and $Q_\circ$ max (Q), then

(1) $Q_\circ$ Δ G.
(2) $Q_\circ$ is S-quasinormal.

Lemma: 2.7 [8]

Suppose T ≤ G and T is NH-embedded. Consider M Δ G

(1) If T ≤ H ≤ G with H subnormal, T is NH-embedded in H.
(2) Consider T a q-group for q ∈ π(G) if M ≤ G, then $^{T}/_{M}$ is NH-embedded in G/M.
(3) M is a q-subgroup, so for any q-subgroup T , $^{TM}/_{M}$ is NH-embedded of $^{G}/_{M}$

Lemma: 2.8 [7,12]

Suppose T ≤ H ≤ K

(1) Consider M Δ K and T to be a prime number group. If T is nearly S-permutable, so TM/M is nearly S-permutable in $^{G}/_{M}$.
(2) Consider T is nearly S-permutable contained in G and T ≤ O_p(G), then T is nearly S-permutable in G.

Lemma: 2.9 [13]

Consider T to be a nearly S-permutable subgroup in G, so T is a prime group for p ∈ π(G) if M Δ G, then T ∩ M is also Nearly S-permutable.

Lemma: 2.10 [14]

Suppose R, S, T contained in a finite group G. So equivalent statements are,

$$R \cap ST = (R \cap S)(R \cap T)$$

$$RS \cap RT = R(S \cap T)$$

3. Proof of theorems

Theorem 1.3

Proof:

We check it conversely, consider that this statement is not true and its example is G. Suppose q is the prime (smallest) which divides the order of G. So, G will be nilpotent we will prove it in the next theorem. Suppose R is a normal and Hall q'-subgroup contained in G. We can check that R follows the results of Lemma 2.1 and Lemma 2.7. R is supersolveable by induction and G consists of S_yl Power property of supersolveable. Now consider P is the prime number (largest) which divides the order of G and P is S_yl(G). So, P Δ G, From Lemma 2.1 and Lemma 2.7 it is clear $^{G}/_{P}$ possesses the hypothesis. Thus $^{G}/_{P}$ is supersolveable.

Suppose M Δ G where M is a minimal normal subgroup. Consider it as $^{G}/_{M}$ satisfied this, $^{G}/_{M}$ is supersolveable. We know that all the supersolveable is saturated formation, M Δ G where M is normal and unique.

So M is the subgroup of P, now we claim M ≤ T T ∈ N (P). From our supposition T is Nearly Hall S-embedded or simply SS-quasinormal.

Now consider if T SS-quasinormal in G.

$$Q \Delta G$$

Moreover, T Δ G from Lemma 2.6 thus M ≤ T.

Consider T is nearly hall S-embedded contained in G. T Δ G such that TH be a hall subgroup and T ∩ H ≤ H. And H = 1, so T = TH is Hall subgroup contained in G.

⇒ T = 1 and so |P| = P

As we know $^{G}/_{P}$ supersolveable, we have G ⇒ supersolveable. This contradiction H ≠ 1 and $M \leq T$. In this way G/H is supersolveable, if $|T \cap H|$ equal to 1 then $|P \cap H|$ is p.

This conclude $M = P \cap H$ with p order, so G is supersolveable.

Again, if $T \cap H$ not equal to 1 check $T \cap H \leq T_{S\in G} \leq T$, and

$$T \cap H = T_{S\in G} \cap H$$

is nearly S-permutable using 2.9 Lemma. Next, we have

$$T \cap H \leq P = O_P(G) \Rightarrow T \cap H$$

Be S-quasinormal or S-permutable using 2.8 Lemma. Using Lemma 2.4 $O'(G) \leq M_G(T \cap H)$.

Note that $T \cap H \Delta G$.

Hence

$$M \leq T \cap H \leq T$$

As required, we have

$$M \leq \cap T = \emptyset(P) \text{ ([6]}$$

$$\emptyset(P) \leq \emptyset(G)$$

And this $M \leq \emptyset(G)$

$G/\emptyset(G)$ supersolveable. This conclude G Supersolveable which contradicts the statement and hence theorem proved.

Theorem 1.4

Proof:

We check conversely the statement is not true and its example is group G with minimal order. Suppose a $S_y l_q$(G) and set of all maximal subgroups of $G_q\{Q_1, Q_2, Q_3, \ldots, Q_n\}$.From the statement of the theorem every element Q_1 of M(G_q) is either Nearly Hall S-embedded subgroup or nearly S-permutable without losing generality, Consider every element of $G_1(G_q) = G_2(G_q) = \{Q_{k+1}, Q_{k+2}, \ldots, Q_n\}$ of M(G_q) is nearly S-permutable for $1 \leq k \leq n$.

This prove can be solved in 5 states as below.

1) $M \Delta G$, where M is minimal and G/M is q-nilpotent. Consider $M \Delta G$ where M is minimal, so $G_q M/M$ is $S_y l_q$(G/M) for only $N/M \in M(G_q M/M)$, suppose Q = N ∩ G_q

   So,

$$N = N \cap G_q M = (N \cap G_q) M = QM$$

   and

$$Q \cap M = (N \cap G_q) \cap M = QM \cap G_q \cap M = G_q \cap M$$

   because

$$|G_q : Q| = |G_q(G_q \cap N) = |G_q N : N| = Q$$

as

Q $\in$ M (G_P) From our supposition Q is either Nearly Hall s-embedded $\infty -$ Nearly s-permutable.

Now let Q is Nearly s-permutable, so $N/M = QM/M$ is also Nearly s-permutable in G/M using Lemma 2.1.

Here we suppose Q is nearly Hall s-embedded contained in G, so $\exists$ $H \Delta G$ s.t QH is a Hall subgroup in G and Q $\cap$ H $\leq$ Q_{SG} clearly $HM/M \Delta G/M$, and

$$QM/M\ HM/M = QHM/M$$

Is a Hall subgroup of G/M as Q $\cap$ M $= G_q \cap$ M is S_yl (M), we have

$$|M \cap QH|_{q'} = |M|_q = |M \cap Q|_q = |(M \cap Q)(M \cap H)|_q$$

$$|M \cap QH|_{q'} = \frac{|QH|_{q'}\,|M|_q}{|MQH|_{q'}} = \frac{|H|_{q'}\,|M|_q}{|MH|_{q'}} = |M \cap Q|_q\ |(M \cap Q)(M \cap H)|_q$$

$$M \cap QH = (M \cap Q)(M \cap H)$$

And thus

$$QM \cap HM = (Q \cap H)\,M$$

Using Lemma 2.11, so

$$\frac{QM}{M} \cap \frac{HM}{M} = \frac{\frac{QM}{M} \cap HM}{M} = \frac{(Q \cap H)M}{M} \leq Q_{SG}M/M$$

As Q_{SG} is nearly S-permutable in G,

So $Q_{SG}M/M$ is nearly S-permutable in G/M using Lemma 2.8

$$Q_{SG}\frac{M}{M} \leq (QM/M)_{\frac{SG}{M}}$$

Thus, $N/M = QM/M$ is nearly Hall S-embedded in G/M.

From all these collections we can say that G/M satisfied the hypothesis. As the choice of G, it is obvious that G/M is p-nilpotent, also $M \Delta G$ and unique minimal. As p-nilpotent class is saturated.

(2) $Q_{q'}(G)$ is equal to 1.

Consider $Q_{q'}(G) \geq 1$, so $M \leq Q_{q'}(G)$ by (1)

by (1) since G/M is p-nilpotent and $G/Q_{q'}(G)$ is p-nilpotent, thus G is also p-nilpotent. This contradicts is our hypothesis. So $Q_{q'}(G) = 1$.

(3) $M \leq Q_i$ for any $Q_i \in M_i(G_q)$ for $T \in M_i(G_q)$, T is nearly Hall s-embedded in G then T Δ G such that TH is Hall subgroup and $T \cap H \leq T_{iG}$. If $H = 1$, so T is a Hall subgroup of G and thus $T = 1$ which implies $|G_q|$ = q, as Q is maximal of G_q. By Burnside Theorem, G is p-nilpotent which is contradicts. Thus $H \neq 1$ and $M \leq H$, if $T \cap M = 1$, then $|M|_q \leq$ q. Hence M is p-nilpotent using Burnside Theorem consider V be normal Hall q'-subgroup of M then $V \Delta G$. By minimality of M $V = 1$. Thus $|M| = q$ consequently, this nilpotency of $G/M \Rightarrow G$ is p-nilpotent which contradicts. So, we have $T \cap M \neq 1$ and $T \cap M \leq T \cap H \leq T_{SG} \leq T$.
We have $T_{SG} \cap M \leq T \cap M$ and thus $T \cap M = T_{SG} \cap M$. Using Lemma 2.9 $T \cap M$ is nearly S-permutable. M is solvable by Lemma 2.10
$\Rightarrow$ M is q'-group and $M \leq Q_q(G)$
Particularly $T \cap M \leq Q_q(G)$ by Lemma 2.8, $T \cap M$ is nearly s-permutable. Hence $O(G) \leq N_G(T \cap M)$ using Lemma 2.4 see $T \cap M \Delta G_q$ which implies $T \cap M \Delta G$.
Hence $T \cap M = M$ it leads to $M \leq T$.

(4) For $Q_j \in M_i(G_q)$, $\exists$ a normal N_j subgroup of G, s.t $M \leq N_j$. For any

$T \in M_i(G_q)$ we have H is ss-quasinormal so $\exists B \le G$ such that $G = TB$ and $TB_q = B_qT$ for every $S_yl_{B_q}(B)$. Thus $|B \; T \cap B|_q = |G\; T|_q = q$ from $G = TB$ so $B_q \notin T$ and $B_qT = B_qT$ is a $S_yl_q(G)$. $H \in M_i(G_q)$ and comparing of orders $|T \cap B|_q = T \cap B$ thus

$$T \bigcap B = \bigcap_{b\in B} (B_q^b \bigcap T) \le \bigcap_{b\in B} B_q^b = O_q(B)$$

From $|O_q(B)B \cap T| = q$ or 1, obtaining $|B/O_q(B)|q = 1$ or q is the smallest prime dividing |G| using Burnside Theorem.

$B/O_q(B)$ is nilpotent B being p-solvable and Hall q'-subgroup of B [1]. Let $H = Hall\ q' - subgroup\ of\ B$, π $H = (P_2, \ldots, P_5), P_i \in S_yL_q(H)$ from ss-quasinormal subgroup definition, T and $< P_2, \ldots, P_5 > = K$ are permutations. So, TH is subgroup of G.

H is Hall q'-subgroup and TH is a subgroup of index q in G. q is smallest prime dividing $|G|, TH\ \Delta\ G\ if\ TH = 1$.

Then G is elementary commutative q-group, which is contradicts. Hence

$$M \le TH = N_j$$

T is supersolvable. Consider p is largest prime that divides order of T and $P = O_q(G)$ char $S_yl_q(T)$, So $P\ \Delta\ T$ contained in G. Clearly $(G/P)/(T/P) \cong G/T$ supersolvable. Using Lemma 2.1 and 2.7 every max S_yl(T/P) is either Nearly Hall s-embedded or nearly S-permutable in G/P. Thus G/P endorse the statement and supersolvable.

(5) Final contradiction:

Take

$$V = (\bigcap_{j=1}^{k} Q_j) \cap (\bigcap_{i=k+1}^{n} N_i)$$

By the statement given above, we know $N \le V$.

We have

$$M = M \cap G_q \le V \cap G_q = \left(\left(\bigcap_{j=1}^{k} Q_j\right) \cap \left(\bigcap_{i=k+1}^{n} N_i\right)\right) \cap G_q = \bigcap_{j=1}^{n} Q_j = \emptyset(G_q)$$

From ([6]. III.3.3) we have $\emptyset(G_q) \le \emptyset(G)$ and thus $M \le \emptyset(G)$. As G/M is p-nilpotent we get $G/\emptyset(G)$ p-nilpotent. We know class of all p-nilpotent is saturated formation, G will be p-nilpotent. Which contradicts and proves the theorem.

Theorem 1.5

Proof:

Consider it false and take a minimal order example to prove. Using Lemma 2.1 and 2.7 we know every max S_yl(T) either nearly Hall s-embedded or nearly s-permutable in H. By Theorem 1.3 consider $M\ \Delta\ G$ where minimal subgroup same as Theorem 1.4, we have G/M satisfies it and supersolvable. We know saturated formation consist of class of all supersolvable group, $M\ \Delta\ G$ will unique minimal contained in P.

Follow the proof of Theorem 1.3, we have $M \le \emptyset(P)$, thus $G/\emptyset(P)$ supersolvable by $([6], iii, 3.3)\ \emptyset(P) \le \emptyset(G)$. Thus $G / \emptyset(G)$ supersolvable. As saturated formation G is supersolvable. If contradict thus complete the proof.

4. Applications

Corollary 4.1 ([5, *Theorem* 3.1]). Suppose q is the smallest prime dividing the order of G and S_yl(G)G_q . Consider every maximal subgroup G_q

is nearly Hall s-embedded in G. Thus, G is nilpotent.

Corollary 4.2 ([1, $Theorem$3.1]). Suppose q is the smallest prime dividing order of G and S_yl(G)G_q . Suppose every maximal G_q subgroup is nearly s-quasigroup subgroup in G. then G is nilpotent.

Corollary 4.3 ([5, $Theorem$ 3.4]). Suppose a $M' \Delta G$ s.t G / E supersolvable. If every maximal of every S_yl(E) is Nearly Hall s-embedded then G is supersolvable.

Data Availability

In this article, no data were utilized.

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